Formal dimension for semisimple symmetric spaces

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Abstract

If G is a semisimple Lie group and $(\pi, \mathcal{H})$ an irreducible unitary representation of G with square integrable matrix coefficients, then there exists a number $d(\pi)$ such that

$$(\forall v, v', w, w' \in \mathcal{H}) \qquad \frac{1}{d(\pi)}\langle v, v'\rangle\langle w', w\rangle = \int_G \langle \pi(g).v, w\rangle\overline{\langle \pi(g).v'.w'\rangle}\ d\mu_G(g).$$

The constant $d(\pi)$ is called the *formal dimension* of $(\pi, \mathcal{H})$ and was computed by Harish-Chandra in [HC56, 66].

If now $H\backslash G$ is a semisimple symmetric space and $(\pi, \mathcal{H})$ an irreducible H-spherical unitary $(\pi, \mathcal{H})$ belonging to the holomorphic discrete series of $H\backslash G$, then one can define a formal dimension $d(\pi)$ in an analogous manner. In this paper we compute $d(\pi)$ for these class of representations.

Keywords: Holomorphic discrete series, highest weight representation, formal dimension, formal degree, spherical representation, c-functions.

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Introduction

Let $H\backslash G$ be a semisimple irreducible simply connected non-compact symmetric space admitting relative holomorphic discrete series, i.e., there exists a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G and a non-zero H-invariant hyperfunction vector $\nu \in \mathcal{H}_\lambda^{-\omega}$ such that

$$\frac{1}{d(\lambda)} := \frac{1}{|\langle \nu, v_\lambda\rangle|^2}\int_{HZ\backslash G} |\langle \nu, \pi_\lambda(g).v_\lambda\rangle|^2\ d\mu_{HZ\backslash G}(HZg)$$

is finite. Here v_λ denotes a highest weight vector, Z the center of G and $\mu_{HZ\backslash G}$ a G-invariant measure on the homogeneous space $HZ\backslash G$. Note that v_λ and ν are unique up to scalar multiple as well as $\langle \nu, v_\lambda\rangle \neq 0$. Therefore the number $d(\lambda)$ is well defined and we call it the *formal dimension* of the spherical highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$. We remark here that our definition of the formal dimension generalizes Harish-Chandra notion in the "group case", i.e., where $G = G_0 \times G_0$ and $H = \Delta(G) = \{(g, g) : g \in G_0\}$ for a simply connected hermitian Lie group G_0 (cf. [HC56] and Remark III.5 below).

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Note that the constants $d(\lambda)$ determine the part of the Plancherel measure on $H\backslash G$ which corresponds to the relative holomorphic discrete series. Thus the explicit knowledge of the formal dimensions gives us a better understanding of the Plancherel Theorem on $H\backslash G$ which was recently obtained by van den Ban-Schlichtkrull and Delorme (cf. [BS97,99], [De98]).

Let $(\mathfrak{g}, \tau)$ be the symmetric Lie algebra attached to $H\backslash G$ and write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the τ-eigenspace decomposition. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a τ-invariant Cartan decomposition of $\mathfrak{g}$, then the algebraic characterization of $H\backslash G$ admitting relative holomorphic discrete series is $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} \neq 0$. Symmetric Lie algebras $(\mathfrak{g}, \tau)$ having this property are called *compactly causal* (cf. [HiÓl96]). In the group case, i.e., $(\mathfrak{g}, \tau) = (\mathfrak{g}_0 \oplus \mathfrak{g}_0, \sigma)$ with $\sigma(X, Y) = (Y, X)$ the flip involution, this just means that $\mathfrak{g}_0$ is hermitian. We remark here that the formal dimension in the group case was computed by Harish-Chandra (cf. [HC56]).

In this paper we derive the formula for the formal dimension $d(\lambda)$ for compactly causal symmetric spaces. For the special class of Cayley type spaces this problem has been dealt with by Chadli with Jordan algebra methods (cf. [Ch98]). The approach presented here is general and purely representation theoretic.

Our key result is the *Averaging Theorem* (cf. Theorem II.16) which asserts that for large parameters λ the H-integral over v_λ converges. More precisely, for large parameters λ we prove that

$$\int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h) = \frac{\langle v_\lambda, v_\lambda \rangle}{\langle \nu, v_\lambda \rangle} c(\lambda + \rho)\nu,$$

where the left hand side has to be understood as a vector valued integral with values in the Fréchet space of hyperfunction vectors and

$$c(\lambda) = \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(\lambda+\rho)} \; d\mu_{\overline{N}}(\overline{n})$$

denotes the c-function of the *non-compactly causal* c-dual space $H^c\backslash G^c$ (cf. [HiÓl96]).

To obtain the formula for the formal degree $d(\lambda)$, we plug in the relation for ν obtained from the Averaging Theorem in the definition of $d(\lambda)$ and obtain for large parameters:

$$d(\lambda) = d(\lambda)^G c(\lambda + \rho),$$

where $d(\lambda)^G$ is the formal dimension of $(\pi_\lambda, \mathcal{H}_\lambda)$ for the relative discrete series on G (cf. Theorem III.6). Using some ideas of Ólafsson and Ørsted (cf. [ÓØ91]) we prove the analytic continuation of our formula for $d(\lambda)$ (cf. Theorem IV.15).

The c-function $c(\lambda)$ can be written as a product

$$c(\lambda) = c_0(\lambda) c_\Omega(\lambda),$$

where $c_0(\lambda)$ is the c-function of a certain Riemannian symmetric subspace of $H\backslash G$ and $c_\Omega(\lambda)$ is the c-function of the real form Ω of the bounded symmetric domain $\mathcal{D} \cong G/K$. In particular we have

$$d(\lambda) = d(\lambda)^G c_0(\lambda + \rho) c_\Omega(\lambda + \rho).$$

The ingredients in this formula of $d(\lambda)$ are known: Harish-Chandra computed $d(\lambda)^G$ in [HC56], Gindikin and Karpelevič $c_0(\lambda)$ (cf. [GiKa62]) and finally Ólafsson and the author computed $c_\Omega(\lambda)$ in [KrÓl99] (see also [Fa95], [Gr97] for earlier results in important special cases).

In the final section we give applications of our results to spherical holomorphic representation theory. Recall that a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G extends naturally to a holomorphic representation of the maximal open complex Ol'shanskiĭ semigroup

$S^0_{\max} = G\,\mathrm{Exp}(iW^0_{\max})$ (cf. [Ne99b, Sect. XI.2]). If $(\pi_\lambda, \mathcal{H}_\lambda)$ is an H-spherical unitary highest weight representation of G, then we define its *spherical character* by

$$\Theta_\lambda: S^0_{\max} \to \mathbb{C}, \quad s \mapsto \frac{\langle v_\lambda, v_\lambda\rangle}{|\langle \nu, v_\lambda\rangle|^2}\langle \pi_\lambda(s).\nu, \nu\rangle.$$

Note that Θ_λ is an H-biinvariant holomorphic function on $S^0_{\max}$. On the other hand on $S^0_{\max} \cap HAN$ one defines the *spherical function with parameter* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ by

$$\varphi_\lambda: S^0_{\max} \cap HAN \to \mathbb{C}, \quad s \mapsto \int_H a_H(sh)^{\lambda-\rho}\, d\mu_H(h),$$

whenever the right hand side makes sense(cf. [FHÓ94] or [KNÓ98]). For large parameters λ we prove the long searched relation of Ólafsson (cf. [Ól98, Open Problem 7(1)])

$$(\forall s \in S^0_{\max} \cap HAN) \quad \Theta_\lambda(s) = \frac{1}{c(\lambda+\rho)}\varphi_{\lambda+\rho}(s)$$

(cf. Theorem V.4). Finally we want to point out that the results of this paper are a major step towards a proof of the *Plancherel Theorem* of G-invariant Hilbert spaces of holomorphic functions on G-invariant subdomains of the Stein variety $\Xi^0_{\max} = G \times_H iW^0_{\max}$ (cf. [Ch98], [HiKr98, 99b], [HÓØ91], [Kr98, 99b], [KNÓ97], [Ne99a].)

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I. Causal symmetric Lie algebras

This subsection is a brief introduction to causal symmetric Lie algebras. Purely algebraic definitions of "causality" are given and the basic notation on the algebraic level is introduced.

Definition I.1. Let $\mathfrak{g}$ denote a finite dimensional Lie algebra over the real numbers.
(a) A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$, where τ is an involutive automorphism of $\mathfrak{g}$. We set

$$\mathfrak{h} := \{X \in \mathfrak{g}: \tau(X) = X\} \quad \text{and} \quad \mathfrak{q} := \{X \in \mathfrak{g}: \tau(X) = -X\}$$

and note that $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. We call $(\mathfrak{g}, \tau)$ *irreducible*, if $\{0\}$ and $\mathfrak{g}$ are the only τ-invariant ideals of $\mathfrak{g}$.
(b) We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. If τ is a involution on $\mathfrak{g}$, we also denote by τ the complex linear extension of τ to a endomorphism of $\mathfrak{g}_{\mathbb{C}}$.
(c) The *c-dual* $\mathfrak{g}^c$ of $(\mathfrak{g}, \tau)$ is defined by $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$.
(d) If $\mathfrak{g}$ is semisimple, then there exists a Cartan involution θ of $\mathfrak{g}$ which commutes with τ (cf. [Be57] or [KrNe96, Prop. I.5(iii)]). We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition. By subscripts we indicate intersections, for example $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$ etc. Since τ and θ commute, we have $\mathfrak{g} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}} + \mathfrak{q}_{\mathfrak{k}} + \mathfrak{q}_{\mathfrak{p}}$. The prescription $\theta^c := \theta\tau|_{\mathfrak{g}^c}$ defines a Cartan involution on $\mathfrak{g}^c$ and we denote by $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$ the corresponding Cartan decomposition of $\mathfrak{g}^c$. ■

If $V \subseteq \mathfrak{g}$ is a subspace, then we set $\mathfrak{z}(V) = \{X \in V: (\forall Y \in V)[X,Y] = 0\}$.

Definition I.2. Let $(\mathfrak{g}, \tau)$ be an irreducible semisimple symmetric Lie algebra and θ a Cartan involution of $\mathfrak{g}$ commuting with τ. Then we call $(\mathfrak{g}, \tau)$

(CC) *compactly causal* if $\mathfrak{z}(\mathfrak{q}_\mathfrak{k}) \neq \{0\}$.

(NCC) *non-compactly causal*, if $(\mathfrak{g}^c, \tau|_{\mathfrak{g}^c})$ is (CC).

(CT) of *Cayley type*, if it is both (CC) and (NCC). ■

Lemma I.3. *Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra. Then the following assertions hold:*

(i) *The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is compactly causal if and only if it belongs to one of the following two types:*

(1) *The Lie algebra $\mathfrak{g}$ is simple hermitian and $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{q}$.*

(2) *The subalgebra $\mathfrak{h}$ is simple hermitian and $(\mathfrak{g}, \tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \sigma)$, where σ denotes the flip involution $\sigma(X, Y) = (Y, X)$.*

(ii) *If $(\mathfrak{g}, \tau)$ is compactly causal, then*

(a) *$\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q}$ is one-dimensional,*

(b) *every maximal abelian subspace $\mathfrak{b} \subseteq \mathfrak{q}_\mathfrak{k}$ is maximal abelian in $\mathfrak{q}$ and $\mathfrak{h}_\mathfrak{p} + \mathfrak{q}_\mathfrak{k}$.*

Proof. (i) This follows from [HiÓl96, Lemma 1.3.5, Th. 1.3.8] or [KrNe96, Prop. V.6].
(ii) This is a consequence of [HiÓl96, Prop. 3.1.11]. ■

Remark I.4. (a) From the view point of convex geometry and complex analysis the compactly causal symmetric spaces are the natural generalization of hermitian groups in the symmetric space setting (cf. [HiÓl96], [KrNe96], [KNÓ97, 98] and [Ne99b]). The compactly and non-compactly causal symmetric Lie algebras have been classified; for a complete list see [HiÓl96, Th. 3.2.8].
(b) Suppose that $H\backslash G$ is a simply connected symmetric space associated to an irreducible semisimple symmetric Lie algebra $(\mathfrak{g}, \tau)$. If $(\mathfrak{g}, \tau)$ is compactly causal, then Lemma I.3(ii)(b) implies that the symmetric space $H\backslash G$ admits relative holomorphic discrete series (cf. [FJ80]). The converse is also true. This result seems to us to be well known. But since we do not know a proof in the literature, we added a proof in Appendix B (cf. Lemma B.1). ■

Let $(\mathfrak{g}, \tau)$ be compactly causal. Recall that this implies in particular that $\mathfrak{g}$ is hermitian (cf. Lemma I.3(i)).

We choose a maximal abelian subalgebra $i\mathfrak{a} \subseteq \mathfrak{q}_\mathfrak{k}$ and extend $i\mathfrak{a}$ in $\mathfrak{k}$ to a compactly embedded Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Recall from Lemma I.3(ii)(b) that $\mathfrak{a}$ is maximal abelian in $i\mathfrak{q}$ and $\mathfrak{p}^c$. Then $\mathfrak{t} = \mathfrak{t}_\mathfrak{h} + i\mathfrak{a}$ and $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} \subseteq i\mathfrak{a}$. By Lemma I.3 we know that $\mathfrak{z}(\mathfrak{k}) \cap \mathfrak{q} = \mathbb{R}Z_0$ is one-dimensional and by [Hel78, Ch. VIII, §7] we can normalize Z_0 in such a way that $\mathrm{Spec}(\mathrm{ad}\, Z_0) = \{-i, 0, i\}$ holds. We denote by $\widehat{\Delta} = \widehat{\Delta}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ the root system of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$ and by $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{a})$ the restricted root system of $\mathfrak{g}^c$ with respect to $\mathfrak{a}$. Note that $\widehat{\Delta}|_\mathfrak{a} \backslash \{0\} = \Delta$. The corresponding root space decompositions are denoted by

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\widehat{\alpha} \in \widehat{\Delta}} \mathfrak{g}_\mathbb{C}^{\widehat{\alpha}} \quad \text{and} \quad \mathfrak{g}^c = \mathfrak{a} \oplus \mathfrak{z}_\mathfrak{h}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in \Delta} (\mathfrak{g}^c)^\alpha.$$

A root $\widehat{\alpha} \in \widehat{\Delta}$ is called *compact* if $\widehat{\alpha}(Z_0) = 0$ and *non-compact* otherwise. Analogously one defines compact and non-compact roots in Δ. Write $\widehat{\Delta}_k$ and $\widehat{\Delta}_n$ for the set of all compact, resp. non-compact, roots in $\widehat{\Delta}$. Analogously one defines Δ_k and Δ_n.

Once and for all we fix now a positive system $\widehat{\Delta}^+$ of $\widehat{\Delta}$ such that

$$\widehat{\Delta}_n^+ := \widehat{\Delta}^+ \cap \widehat{\Delta}_n = \{\widehat{\alpha} \in \widehat{\Delta}_n : \widehat{\alpha}(Z_0) = i\}$$

holds. A positive system Δ^+ of Δ is defined by $\Delta^+ := \widehat{\Delta}^+|_\mathfrak{a} \backslash \{0\}$.

II. Spherical highest weight representations

In this section we briefly recall the classification of analytic and hyperfunction vectors of a a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of a simply connected compactly causal group (G, τ). Further we collect the basic facts of H-spherical highest weight representations. Then, after giving the definitions of the various c-functions associated to the non-compactly causal c-dual space (G^c, τ) of (G, τ), we prove the key result of the whole paper: The Averaging Theorem, which asserts that for large parameters λ the H-integral over the highest weight vector converges in the Fréchet space of hyperfunction vectors. One obtains the up to scalar multiple uniquely determined H-spherical vector with a normalization constant which is given by a certain c-function.

Unitary highest weight representations

Recall that if G is a simply connected Lie group associated to a symmetric Lie algebra $(\mathfrak{g}, \tau)$, then τ integrates to an involution on G, also denoted by τ, and that the fixed point group G^τ is connected (cf. [Lo69, Th. 3.4].)

To a compactly causal symmetric Lie algebra $(\mathfrak{g}, \tau)$ we associate the following analytic objects:

G	simply connected Lie group with Lie algebra $\mathfrak{g}$,
G^c	simply connected Lie group with Lie algebra $\mathfrak{g}^c$,
$G_\mathbb{C}$	simply connected Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$,
H	τ-fixed points in G,
H^c	τ-fixed points in G^c,
$H_\mathbb{C}$	τ-fixed points in $G_\mathbb{C}$,
K	analytic subgroup in G corresponding to $\mathfrak{k}$,
K^c	analytic subgroup in G^c corresponding to $\mathfrak{k}^c$,
$K_\mathbb{C}$	analytic subgroup in $G_\mathbb{C}$ corresponding to $\mathfrak{k}_\mathbb{C}$,
H^0	centralizer of $\mathfrak{a}$ in H,
$H^{c,0}$	centralizer of $\mathfrak{a}$ in H^c,
Z	center of G.

Note that even though both H and H^c are connected and have the same Lie algebra, they are in general not isomorphic. Recall that $Z \subseteq K$.

If X is a topological space and $Y \subseteq X$ is a subspace, then we denote by Y^0 or $\operatorname{int} Y$ the interior of Y in X.

For each $X \in \mathfrak{g}_\mathbb{C}$ we denote by $\overline{X}$ the complex conjugate of X with respect to the real form $\mathfrak{g}$.

Definition II.1. (Complex Ol'shanskiĭ semigroups, cf. [Ne99, Ch. XI]) Let $(\mathfrak{g}, \tau)$ be a compactly causal symmetric Lie algebra and $\widehat{\Delta}^+ = \widehat{\Delta}^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ be the positive system from Section I.

(a) Associated to $\widehat{\Delta}^+$ we define the *maximal cone* in $\mathfrak{t}$ by

$$\widehat{C}_{\max} = \{X \in \mathfrak{t} \colon (\forall \alpha \in \widehat{\Delta}_n^+)\ i\alpha(X) \geq 0\}.$$

We set $\widehat{W}_{\max} := \overline{\mathrm{Ad}(G).\widehat{C}_{\max}}$ and note that $\widehat{W}_{\max}$ is a closed convex $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}$ admitting no affine lines and which is maximal with respect to these properties (cf. [Ne99b, Sect. VII.3]).

(b) Let $G_1 := \langle \exp_{G_{\mathbb{C}}}(\mathfrak{g}) \rangle$. By Lawson's Theorem $S_{\max,1} := G_1 \exp_{G_{\mathbb{C}}}(i\widehat{W}_{\max})$ is a closed subsemigroup of $G_{\mathbb{C}}$, the *maximal complex Ol'shanskiĭ semigroup*, and the *polar map*

$$G_1 \times \widehat{W}_{\max} \to S_{\max,1}, \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism (cf. [La94, Th. 3.4]).

Denote by $S_{\max}$ the universal covering semigroup of $S_{\max,1}$ and write $\mathrm{Exp}\colon \mathfrak{g} + i\widehat{W}_{\max} \to S_{\max}$ for the lifting of $\exp_{G_{\mathbb{C}}} \mid_{\mathfrak{g}+i\widehat{W}_{\max}} \colon \mathfrak{g} + i\widehat{W}_{\max} \to S_{\max,1}$. Then it is easy to see that $S_{\max} = G\,\mathrm{Exp}(i\widehat{W}_{\max})$ and that there is a polar map

$$G \times \widehat{W}_{\max} \to S, \quad (g, X) \mapsto g\,\mathrm{Exp}(iX)$$

which is homeomorphism. We define the *interior* of $S_{\max}$ by $S^0_{\max} := G\,\mathrm{Exp}(i\widehat{W}^0_{\max})$. Note that $S^0_{\max}$ is an open semigroup ideal in $S_{\max}$ which carries a natural complex structure for which the semigroup multiplication is holomorphic. Further the prescription $s = g\,\mathrm{Exp}(iX) \mapsto s^* = \mathrm{Exp}(iX)g^{-1}$ defines on $S_{\max}$ the structure of an *involutive semigroup.* Note that the involution is antiholomorphic on $S^0_{\max}$. ■

Remark II.2. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G with respect to $\widehat{\Delta}^+$ and highest weight $\lambda \in i\mathfrak{t}^*$. Denote by $B(\mathcal{H}_\lambda)$ the C^*-algebra of bounded operators on $\mathcal{H}_\lambda$. Recall from [Ne99b, Th. XI.4.8] that $(\pi_\lambda, \mathcal{H}_\lambda)$ has a natural extension to a *holomorphic representation* $\pi_\lambda\colon S_{\max} \to B(\mathcal{H}_\lambda)$ of $S_{\max}$, i.e., π_λ is strongly continuous, holomorphic when restricted to $S^0_{\max}$ and satisfies $\pi_\lambda(s^*) = \pi_\lambda(s)^*$ for all $s \in S_{\max}$.

Note that for $X \in \widehat{W}_{\max}$ one has $\pi_\lambda(\mathrm{Exp}(iX)) = e^{id\pi_\lambda(X)}$.

■

Definition II.3. Let G be a Lie group and $\mathcal{H}$ a Hilbert space. If $(\pi, \mathcal{H})$ is a unitary representation of G, then we call $v \in \mathcal{H}$ an *analytic vector* if the orbit map $G \to \mathcal{H},\ g \mapsto \pi(g).v$ is analytic. We denote by $\mathcal{H}^\omega$ the vector space of all analytic vectors of $(\pi, \mathcal{H})$. There is a natural locally convex topology on $\mathcal{H}^\omega$ for which the representation $(\pi^\omega, \mathcal{H}^\omega)$ of G on $\mathcal{H}^\omega$ is continuous (cf. [KNÓ97, Appendix]). The strong antidual of $\mathcal{H}^\omega$ is denoted by $\mathcal{H}^{-\omega}$ and the elements of $\mathcal{H}^{-\omega}$ are called *hyperfunction vectors.* Note that there is a natural chain of continuous inclusions

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}.$$

The natural extension of $(\pi, \mathcal{H})$ to a representation on the space of hyperfunction vectors is denoted by $(\pi^{-\omega}, \mathcal{H}^{-\omega})$ and given explicitly by

$$\langle \pi^{-\omega}(g).\nu, v \rangle := \langle \nu, \pi^\omega(g^{-1}).v \rangle.$$ ■

Proposition II.4. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G with respect to $\widehat{\Delta}^+$ and highest weight λ. Let $X \in \operatorname{int} \widehat{W}_{\max}$ be an arbitrary element. Then the analytic vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$ are given by*

$$\mathcal{H}^\omega_\lambda = \bigcup_{t>0} \pi_\lambda(\mathrm{Exp}(tiX)).\mathcal{H}_\lambda$$

and the topology on $\mathcal{H}^\omega_\lambda$ is the finest locally convex topology making for all $t > 0$ the maps $\mathcal{H}_\lambda \to \mathcal{H}^\omega_\lambda,\ v \mapsto \pi_\lambda(\mathrm{Exp}(tiX)).v$ continuous.

Proof. [KNÓ98, Prop. A.5]. ■

If $\lambda \in i\mathfrak{t}^*$ is dominant integral for $\widehat{\Delta}_k^+$, we denote by $(\pi_\lambda^K, F(\lambda))$ the irreducible highest weight representation of K with highest weight λ. Note that $(\pi_\lambda^K, F(\lambda))$ extends naturally to a holomorphic representation of the universal covering group $\widetilde{K_\mathbb{C}}$ of $K_\mathbb{C}$ and which we denote by the same symbol.

Remark II.5. Recall that $(\pi_\lambda, \mathcal{H}_\lambda)$ can be realized in the Fréchet space $\mathrm{Hol}(\mathcal{D}, F(\lambda))$ of $F(\lambda)$-valued holomorphic functions on the Harish-Chandra realization $\mathcal{D}$ of the hermitian symmetric space G/K. So let us assume $\mathcal{H}_\lambda \subseteq \mathrm{Hol}(\mathcal{D}, F(\lambda))$. Then for all $z \in \mathcal{D}$ and $v \in F(\lambda)$ the point evaluation

$$\mathcal{H}_\lambda \to \mathbb{C}, \quad f \mapsto \langle f(z), v\rangle$$

is continuous, hence can be written as $\langle f(z), v\rangle = \langle f, K_{z,v}^\lambda\rangle$ for some $K_{z,v}^\lambda \in \mathcal{H}_\lambda$. One can show that all vectors $K_{z,v}^\lambda$ are analytic. Then, if $\overline{\mathcal{H}_\lambda}$ denotes the closure of $\mathcal{H}_\lambda$ in the nuclear Fréchet space $\mathrm{Hol}(\mathcal{D}, F(\lambda))$, then the mapping

$$r\colon \mathcal{H}_\lambda^{-\omega} \to \mathrm{Hol}(\mathcal{D}, F(\lambda)), \quad \nu \mapsto r(\nu); \ \langle r(\nu)(z), v\rangle = \nu(K_{z,v}^\lambda)$$

is a G-equivariant topological isomorphism onto its closed image $\operatorname{im} r = \overline{\mathcal{H}_\lambda}$. In particular $\mathcal{H}_\lambda^{-\omega}$ is a nuclear Fréchet space (cf. [Kr99a, Sect. III] for all that). ■

Spherical representations

Definition II.6. Let G be a Lie group, $H \subseteq G$ a closed subgroup and $(\pi, \mathcal{H})$ a unitary representation of G. Then we write $(\mathcal{H}^{-\omega})^H$ for the set of all those elements $\nu \in \mathcal{H}^{-\omega}$ satisfying $\pi^{-\omega}(h).\nu = \nu$ for all $h \in H$. The unitary representation $(\pi, \mathcal{H})$ is called H-*spherical* if there exists a cyclic vector $\nu \in (\mathcal{H}^{-\omega})^H$ for $(\pi^{-\omega}, \mathcal{H}^{-\omega})$. ■

For $\lambda \in i\mathfrak{t}^*$ dominant integral with respect to $\widehat{\Delta}_k^+$ recall the definition of the *generalized Verma module*

$$N(\lambda) := \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C} \ltimes \mathfrak{p}^+)} F(\lambda)$$

which is a highest weight module of $\mathfrak{g}$ with respect to $\widehat{\Delta}^+$ and highest weight λ (cf. [EHW83]). We denote by $L(\lambda)$ the unique irreducible quotient of $N(\lambda)$.

Proposition II.7. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G with respect to $\widehat{\Delta}^+$.*

(i) *If $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical, then $(\pi_\lambda^K, F(\lambda))$ is $H \cap K$-spherical. In particular $\lambda \in \mathfrak{a}^*$ and the highest weight vector $v_\lambda \in \mathcal{H}_\lambda$ is fixed by H^0.*

(ii) *The restriction mapping*

$$\mathrm{Res}\colon (\mathcal{H}^{-\omega})^H \to F(\lambda)^{H\cap K}, \quad \nu \mapsto \nu|_{F(\lambda)}$$

is injective. In particular, $\dim(\mathcal{H}^{-\omega})^H \leq 1$ and $\langle \nu, v_\lambda\rangle \neq 0$ for $\nu \neq 0$. Moreover if $L(\lambda) = N(\lambda)$, then Res *is a bijection, i.e., $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical if and only if $(\pi_\lambda^K, F(\lambda))$ is $H \cap K$-spherical.*

Proof. (i) is a special case of [KNÓ97, Prop. VI.5] and (ii) a special case of [Kr99a, Th. III.14]. ■

Remark II.8. In general it is not true that $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical if the minimal K-type $(\pi_\lambda^K, F(\lambda))$ is $H \cap K$-spherical. For a counter example see [Kr99a, Ex. III.16]. ■

The c-functions on the c-dual space $H^c\backslash G^c$.

To the positve system $\Delta^+ = \Delta^+(\mathfrak{g}^c, \mathfrak{a})$ we associate several subalgebras of $\mathfrak{g}^c$

$$\mathfrak{n} = \bigoplus\nolimits_{\alpha\in\Delta^+} (\mathfrak{g}^c)^\alpha, \qquad \overline{\mathfrak{n}} = \bigoplus\nolimits_{\alpha\in\Delta^-} (\mathfrak{g}^c)^\alpha,$$

$$\mathfrak{n}_n^\pm = \bigoplus\nolimits_{\alpha\in\Delta_n^\pm} (\mathfrak{g}^c)^\alpha, \qquad \mathfrak{n}_k^\pm = \bigoplus\nolimits_{\alpha\in\Delta_k^\pm} (\mathfrak{g}^c)^\alpha.$$

Further we set

$$\mathfrak{p}^\pm := \bigoplus_{\widehat{\alpha}\in\widehat{\Delta}_n^+} \mathfrak{g}_{\mathbb{C}}^{\widehat{\alpha}} \quad \text{and} \quad \mathfrak{g}(0) := \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{k}} \subseteq \mathfrak{g}^c.$$

Remark II.9. (a) The subalgebras $\mathfrak{p}^\pm$ and $\mathfrak{k}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ are invariant under complex conjugation with respect to $\mathfrak{g}^c$ and we have $\mathfrak{p}^\pm \cap \mathfrak{g}^c = \mathfrak{n}_n^\pm$ as well as $\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}^c = \mathfrak{g}(0)$. Thus the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$ induces a splitting in subalgebras of $\mathfrak{g}^c$

$$\mathfrak{g}^c = \mathfrak{n}_n^+ \oplus \mathfrak{g}(0) \oplus \mathfrak{n}_n^-.$$

(b) Recall that $\mathfrak{g}^c = \mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The $\mathfrak{h} \oplus \mathfrak{a} \oplus \mathfrak{n}$- decomposition restricted to $\mathfrak{g}(0)$ coincides with an Iwasawa decomposition of $\mathfrak{g}(0)$ given by $\mathfrak{g}(0) = \mathfrak{k}(0) \oplus \mathfrak{a} \oplus \mathfrak{n}_k^+$, where $\mathfrak{k}(0) := \mathfrak{h} \cap \mathfrak{g}(0) = \mathfrak{k}^c \cap \mathfrak{g}(0)$. ∎

We let $H_{\mathbb{C}} \cap K_{\mathbb{C}}$ act on $H_{\mathbb{C}} \times K_{\mathbb{C}}$ from the left by $x.(h,k) := (hx^{-1}, xk)$ and denote by

$$M := H_{\mathbb{C}} \times_{H_{\mathbb{C}}\cap K_{\mathbb{C}}} K_{\mathbb{C}}$$

the corresponding quotient space. The $H_{\mathbb{C}} \cap K_{\mathbb{C}}$-coset of an element $(h,k) \in K_{\mathbb{C}} \times H_{\mathbb{C}}$ is denoted by $[h,k]$. If $\widetilde{H_{\mathbb{C}}}$ and $\widetilde{K_{\mathbb{C}}}$ denote the universal coverings of $H_{\mathbb{C}}$ and $K_{\mathbb{C}}$, respectively, then we realize the universal cover $\widetilde{M}$ of M by

$$\widetilde{M} = \widetilde{H_{\mathbb{C}}} \times_{(\widetilde{H_{\mathbb{C}}}\cap\widetilde{K_{\mathbb{C}}})_0} \widetilde{K_{\mathbb{C}}}.$$

Further let $P^\pm := \exp_{G_{\mathbb{C}}}(\mathfrak{p}^\pm)$. Recall that $\mathfrak{p}^\pm$ are abelian and that the exponential mapping $\exp_{G_{\mathbb{C}}}|_{\mathfrak{p}^\pm} : \mathfrak{p}^\pm \to P^\pm$ is an isomorphism. In particular $P^\pm$ is simply connected.

Proposition II.10. (The $H_{\mathbb{C}}K_{\mathbb{C}}P^+$-decomposition) *The following assertions hold:*

(i) *The multiplication mapping*

$$M \times P^+ \to G_{\mathbb{C}}, \quad ([h,k], p_+) \mapsto hkp_+$$

is a biholomorphic map onto its open image $H_{\mathbb{C}}K_{\mathbb{C}}P^+$. Furthermore:

(a) *The open submanifold $H_{\mathbb{C}}K_{\mathbb{C}}P^+$ is dense in $G_{\mathbb{C}}$ with complement of Haar measure zero.*

(b) *We have $S_{\max,1} \subseteq H_{\mathbb{C}}K_{\mathbb{C}}P^+$.*

(ii) *If $j : S_{\max,1} \to M \times P^+$ denotes the injection obtained from the isomorphism in* (i)*, then j lifts to an inclusion mapping $\widetilde{j} : S_{\max} \to \widetilde{M} \times P^+$.*

Proof. (i) [KNÓ97, Prop. II.6, Lemma III.7].
(ii) Since $\pi_1(S_{\max,1}) = \pi_1(G_1) \subseteq Z(G) \subseteq Z(K)$, it suffices to show that $\widetilde{j}\,|_K$ is injective. We may assume that $K \subseteq \widetilde{K_{\mathbb{C}}}$, since both K and $\widetilde{K_{\mathbb{C}}}$ are simply connected and $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{k}_{\mathbb{C}}$. Further K normalizes P^+, and so establishing the injectivity of $\widetilde{j}\,|_K$ boils down to proving injectivity of $K \to \widetilde{M},\ k \mapsto [\mathbf{1}, k]$, which is obvious. ∎

We denote by $G(0)$, A, N, $\overline{N}$, $N_k^\pm$ and $N_n^\pm$ the analytic subgroups of G^c corresponding to $\mathfrak{g}(0)$, $\mathfrak{a}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$, $\mathfrak{n}_k^\pm$ and $\mathfrak{n}_n^\pm$.

Remark II.11. (a) In view of the Bruhat decomposition of $\widetilde{K_{\mathbb{C}}}$, we may identify AN_k^+ as a subgroup of $\widetilde{K_{\mathbb{C}}}$. Note that $N = N_k^+ \rtimes N_n^+$ and so every $n \in N$ can be written uniquely as $n = n_k n_n$ with $n_k \in N_k^+$ and $n_n \in N_n^+$. Thus we conclude from Proposition II.10(ii) that the map

$$H \times A \times N \to \widetilde{M} \times P^+, \quad (h, a, n_k n_n) \mapsto ([h, an_k], n_n)$$

is an analytic diffeormphism onto its image which we denote by HAN. Accordingly every element $s \in HAN$ can be written uniquely as $s = h_H(s) a_H(s) n_H(s)$ with $h_H(s) \in H$, $a_H(s) \in A$ and $n_H(s) \in N$ all depending analytically on $s \in HAN$.
(b) If $\mathcal{D} \subseteq \mathfrak{p}^+$ denotes the Harish-Chandra realization of the hermitian symmetric space G/K and $\overline{\mathcal{D}}$ its conjugate in $\mathfrak{p}^-$, then we set $\Omega := \overline{\mathcal{D}} \cap \mathfrak{n}_n^-$. In the sequel we realize Ω as a subset of N_n^- via the exponential mapping. Recall from [KNÓ98, Lemma I.18] that

$$H^c AN = \Omega G(0) N_n^+ \quad \text{and} \quad \overline{N} \cap H^c AN = \Omega \rtimes N_k^- .$$

On the other hand Ω can also naturally be realized in $\widetilde{M} \times P^+$. In particular we obtain that the submanifold $\Omega \rtimes N_k^-$ of $\overline{N}$ is naturally included in $\widetilde{M} \times P^+$. Denote this realization by $\overline{N} \cap HAN$. Further the HAN-decomposition and the $H^c AN$-decomposition (cf. [KNÓ97, Prop. II.4]) coincide on $\overline{N} \cap HAN$. In the sequel we will use this fact frequently without mentioning it.
(c) Let $p: X \to H^c AN$ the universal covering of $H^c AN$. Since X is simply connected, there exists a natural regular map $\pi: X \to \widetilde{M} \times P^+$ with $\pi(X) = HAN$. In particular, the prescription

$$K^c \cap HAN := \pi(p^{-1}(K^c \cap H^c AN))$$

defines an open submanifold of HAN. ■

Note that the exponential mapping $\exp_{\widetilde{K_{\mathbb{C}}}} |_{\mathfrak{a}} : \mathfrak{a} \to A$ is an isomorphism, hence has an inverse which we denote by $\log: A \to \mathfrak{a}$. For each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$ we set $a^\lambda = e^{\lambda(\log a)}$.

Definition II.12. (The c-functions) We write ρ, ρ_k and ρ_n for the elements of $\mathfrak{a}^*$ given by $\frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{n}}$, $\frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{n}_k^+}$ and $\frac{1}{2} \operatorname{tr} \operatorname{ad}_{\mathfrak{n}_n^+}$, respectively. To $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we associate the following *c-functions*:

$$c(\lambda) := \int_{\overline{N} \cap (HAN)} a_H(\overline{n})^{-(\lambda+\rho)} \, d\mu_{\overline{N}}(\overline{n}),$$

$$c_\Omega(\lambda) := \int_\Omega a_H(\overline{n})^{-(\lambda+\rho)} \, d\mu_{N_n^-}(\overline{n}),$$

and

$$c_0(\lambda) := \int_{N_k^-} a_H(\overline{n})^{-(\lambda+\rho_k)} \, d\mu_{N_k^-}(\overline{n})$$

provided the integrals converge absolutely (cf. [FHÓ94] and [KNÓ98]). We write $\mathcal{E}$ for the set of all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which the defining integral for c converges absolutely. Accordingly we define $\mathcal{E}_\Omega$ and $\mathcal{E}_0$. Note that c_0 is the c-function of the non-compact Riemannian symmetric space $K(0) \backslash G(0)$, where $K(0) := G(0)^\tau$. ■

For each $\alpha \in \Delta^+$ let $\check{\alpha} \in \mathfrak{a}$ be the corresponding *coroot*, i.e., $\check{\alpha} \in [(\mathfrak{g}^c)^\alpha, (\mathfrak{g}^c)^{\tau\alpha}]$ such that $\alpha(\check{\alpha}) = 2$. Associated to Δ^+ we define two minimal cones in $\mathfrak{a}$ by

$$C_{\min} := \operatorname{cone}(\{\check{\alpha} : \alpha \in \Delta_n^+\}) \quad \text{and} \quad \check{C}_k := \operatorname{cone}(\{\check{\alpha} : \alpha \in \Delta_k^+\}).$$

Definition II.13. Let V be a finite dimensional vector space and V^* its dual.
(a) If $C \subseteq V$ is a convex set, then its *limit cone* is defined by $\lim C = \{x \in V : x + C \subseteq C\}$. Note that $\lim C$ is a convex cone and that $\lim C$ is closed if C is open or closed.
(b) If $E \subseteq V$ is a subset, then its *dual cone* is defined by $E^\star := \{\alpha \in V^* : \alpha|_V \geq 0\}$. Note that $E^\star$ is a closed convex cone in V^*. ■

Theorem II.14. *The various c-functions are related by*

$$c(\lambda) = c_0(\lambda) c_\Omega(\lambda)$$

and $\mathcal{E} = \mathcal{E}_\Omega \cap \mathcal{E}_0$. Further:

(i) *The domain of convergence $\mathcal{E}_\Omega$ of c_Ω is a tube domain $\mathcal{E}_\Omega = i\mathfrak{a}^* + \mathcal{E}_{\Omega,\mathbb{R}}$ with*

$$\mathcal{E}_{\Omega,\mathbb{R}} = \{\lambda \in \mathfrak{a}^* \colon (\forall \alpha \in \Delta_n^+)\ \lambda(\check{\alpha}) < 2 - m_\alpha\},$$

where $m_\alpha := \dim(\mathfrak{g}^c)^\alpha$. Further for all $\lambda \in \mathcal{E}_\Omega$ we have

$$c_\Omega(\lambda) = \prod_{\alpha \in \Delta_n^+} B\Big(-\frac{\lambda(\check{\alpha})}{2} - \frac{m_\alpha}{2} + 1, \frac{m_\alpha}{2}\Big),$$

where B denotes the Euler Beta-function. In particular:

(a) $-\rho - C_{\min}^\star \subseteq \mathcal{E}_{\Omega,\mathbb{R}}$ *and* $\lim \mathcal{E}_{\Omega,\mathbb{R}} = -C_{\min}^\star$.

(b) *The function c_Ω is holomorphic on $\mathcal{E}_\Omega$ and $c_\Omega|_{\mathcal{E}_\Omega + \mu}$ is bounded for all $\mu \in -\rho - C_{\min}^\star$.*

(ii) *The domain of convergence of c_0 is given by*

$$\mathcal{E}_0 = i\mathfrak{a}^* + \operatorname{int} \check{C}_k^\star,$$

c_0 is holomorphic on $\mathcal{E}_0$ and $c_0|_{\mathcal{E}_0 + \mu}$ is bounded for all $\mu \in \rho_k + \check{C}_k^\star$.

Proof. The product formula $c(\lambda) = c_0(\lambda) c_\Omega(\lambda)$ and the relation $\mathcal{E} = \mathcal{E}_\Omega \cap \mathcal{E}_0$ are a special case of [KNÓ98, Lemma IV.5].
(i) [KrÓl99, Th. III.5].
(ii) All this follows from the Gindikin-Karpelevic product formula for c_0 (cf. [Hel84, Ch. IV, Th. 6.13]). ■

The Averaging Theorem

Lemma II.15. *The group H^0 is compact and up to normalization of Haar measures for all $f \in L^1(H/H^0)$ the following integration formulas hold:*

(i)

$$\int_H f(hH^0)\ d\mu_H(h) = \int_{\overline{N} \cap (HAN)} f(h_H(\overline{n}) H^0) a_H(\overline{n})^{-2\rho}\ d\mu_{\overline{N}}(\overline{n}).$$

(ii)

$$\int_H f(hH^0)\ d\mu_H(h) = \int_{K^c \cap (HAN)} f(h_H(k) H^0) a_H(k)^{-2\rho}\ d\mu_{K^c}(k).$$

Proof. In [KNÓ98, Lemma III.15(i)] it is proved that $H^{c,0}$ is compact and exactly the same argument also yields that H^0 is compact. In view of this fact and our identifications of the various domains in the big complex manifold $\widetilde{M} \times P^+$ (cf. Remark II.11), (i) follows from [KNÓ98, Prop. 1.19] and (ii) from [Ól87, Lemma 1.3]. ■

Theorem II.16. (The Averaging Theorem) *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a unitary highest weight representation of G for which $(\pi_\lambda^K, F(\lambda))$ is $H \cap K$-spherical. If v_λ is a highest weight vector, then the vector valued integral*

$$\int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h)$$

with values in the Fréchet space $\mathcal{H}_\lambda^{-\omega}$ (cf. Remark II.5*) converges and defines a non-zero H-fixed hyperfunction vector if and only if $\lambda+\rho \in \mathcal{E}_\Omega$. If this condition is satisfied and $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$, then*

$$\int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h) = \frac{\langle v_\lambda, v_\lambda\rangle}{\langle \nu, v_\lambda\rangle} c(\lambda+\rho)\nu.$$

Proof. Step 1: The analytic function $S^0_{\max} \cap HAN \to \mathcal{H}_\lambda$, $s \mapsto \pi_\lambda(s).v_\lambda$ extends to an analytic function $F\colon HAN \to \mathcal{H}_\lambda$ and is given ecplicitly by $F(s) = a_H(s)^\lambda \pi_\lambda(h_H(s)).v_\lambda$.

In fact since $d\pi_\lambda(X).v_\lambda = 0$ for all $X \in \mathfrak{n}$, the standard argument of differentiating yields

$$\pi_\lambda(s).v_\lambda = \pi_\lambda(h_H(s)a_H(s)n_H(s)).v_\lambda = \pi_\lambda(h_H(s)a_H(s)).v_\lambda = a_H(s)^\lambda \pi_\lambda(h_H(s)).v_\lambda,$$

establishing Step 1.

Step 2: The integral exists if and only if $\lambda+\rho \in \mathcal{E}_\Omega$.

Let $X \in \operatorname{int} \widehat{W}_{\max}$ be an arbitrary element and set $a_t := \operatorname{Exp}(itX)$ for all $t > 0$. For each $t > 0$ consider the possibly unbounded linear functional

$$f_t\colon \mathcal{H}_\lambda \to \mathbb{C}, \quad w \mapsto \int_H \langle \pi_\lambda(h).v_\lambda, \pi_\lambda(a_t).w\rangle \; d\mu_H(h).$$

In view of Proposition II.4, we have to show that $\lambda+\rho \in \mathcal{E}_\Omega$ is equivalent to $f_t \in \mathcal{H}_\lambda'$ for all $t > 0$.

Since v_λ is fixed by H^0 (cf. Proposition II.7(i)), Step 1 and the integration formula of Lemma II.15(ii) yield

$$\begin{aligned}
&\int_H \langle \pi_\lambda(h).v_\lambda, \pi_\lambda(a_t).w\rangle \; d\mu_H(h)\\
&\quad= \int_{K^c\cap(HAN)} \langle \pi_\lambda(h_H(k)).v_\lambda, \pi_\lambda(a_t).w\rangle a_H(k)^{-2\rho} \; d\mu_{K^c}(k)\\
&\quad= \int_{K^c\cap(HAN)} \langle \pi_\lambda(ka_H(k)^{-1}).v_\lambda, \pi_\lambda(a_t).w\rangle a_H(k)^{-2\rho} \; d\mu_{K^c}(k)\\
&\quad= \int_{K^c\cap(HAN)} \langle \pi_\lambda(a_tk).v_\lambda, w\rangle a_H(k)^{-(\lambda+2\rho)} \; d\mu_{K^c}(k).
\end{aligned} \tag{2.1}$$

Recall from [FHÓ94, Prop. 5.3] that

$$\mathcal{E}_\Omega = \{\lambda \in \mathfrak{a}_\mathbb{C}^*\colon \int_{K^c\cap(HAN)} a_H(k)^{-\operatorname{Re}(\lambda+\rho)} \; d\mu_{K^c}(k) < \infty\}. \tag{2.2}$$

In view of [KNÓ98, Lemma III.15(ii)], the set $X_t := \overline{a_t(K^c \cap HAN)}$ is a compact subset of HAN. In particular we find compact sets C_H^t, C_A^t, C_N^t contained in H, A and N, respectively, such that $X_t \subseteq C_H^t C_A^t C_K^t$. Thus we conclude from Step 1 that

$$(2.3) \qquad (\forall w \in \mathcal{H}_\lambda)(\forall x \in X_t) \quad |\langle \pi_\lambda(x).v_\lambda, w\rangle| \leq \sup_{a \in C_A^t} a^\lambda \|v_\lambda\| \cdot \|w\| < \infty.$$

Hence, in view of (2.1), (2.2) and (2.3) the proof of Step 2 will be complete, provided we can show that for each element x in the compact space X_t we can find an open neighborhood $U \subseteq X_t$ of x and an element $w \in \mathcal{H}_\lambda$ such that $\inf_{y\in U} |\langle \pi_\lambda(y).v_\lambda, w\rangle| > 0$ holds. But this follows from $\langle \pi_\lambda(y).v_\lambda, w\rangle = \langle F(y), w\rangle$ and the continuity of F.

Step 3: If the integral exists, then its value is $\frac{\langle v_\lambda, v_\lambda\rangle}{\langle \nu, v_\lambda\rangle} c(\lambda+\rho)\nu$.

By Step 1 we know that $\lambda+\rho \in \mathcal{E}_\Omega$ in the case where the integral exists. Since λ is a highest weight for an $H\cap K$-spherical representation of K, it has to be dominant integral with respect to Δ_k^+, i.e., $\langle \lambda, \alpha\rangle \in \mathbb{N}_0$ for all $\alpha \in \Delta_k^+$. In particular $c(\lambda+\rho)$ exists (cf. Theorem II.14). Now by Step 2, we know that $\int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h) \in (\mathcal{H}_\lambda^{-\omega})^H$. Since $\dim(\mathcal{H}_\lambda^{-\omega})^H \leq 1$ (cf. Proposition II.7(ii)), it follows that $\int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h) = c\nu$ for some constant $c \in \mathbb{C}$. To determine c we apply the integral to the element v_λ. With Step 1 and the integration formula of Lemma II.15(i) we compute

$$\begin{aligned}
&\int_H \langle \pi_\lambda(h).v_\lambda, v_\lambda\rangle \; d\mu_H(h) \\
&\quad = \int_{\overline{N}\cap(HAN)} \langle \pi_\lambda(h_H(\overline{n})).v_\lambda, v_\lambda\rangle a_H(\overline{n})^{-2\rho} \; d\mu_{\overline{N}}(\overline{n}) \\
&\quad = \int_{\overline{N}\cap(HAN)} \langle \pi_\lambda(\overline{n} a_H(\overline{n})^{-1}).v_\lambda, v_\lambda\rangle a_H(\overline{n})^{-2\rho} \; d\mu_{\overline{N}}(\overline{n}) \\
&\quad = \int_{\overline{N}\cap(HAN)} \langle \pi_\lambda(\overline{n}).v_\lambda, v_\lambda\rangle a_H(\overline{n})^{-(\lambda+2\rho)} \; d\mu_{\overline{N}}(\overline{n}) \\
&\quad = \langle v_\lambda, v_\lambda\rangle \int_{\overline{N}\cap(HAN)} a_H(\overline{n})^{-(\lambda+2\rho)} \; d\mu_{\overline{N}}(\overline{n}) \\
&\quad = \langle v_\lambda, v_\lambda\rangle c(\lambda+\rho).
\end{aligned}$$

This proves Step 3 and completes the proof of the theorem. ■

III. Representations of the relative discrete series

In this section we state and prove the Harish-Chandra - Godement Orthogonality relations for homogeneous spaces carrying an invariant measure. Then we give the definition of the formal dimension $d(\lambda)$ of a unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ which belongs to the relative discrete series of $H\backslash G$. Finally we derive the formula for $d(\lambda)$ for large values of λ.

Orthogonality Relations

Definition III.1. Let G be a Lie group, Z its center and $\widehat{Z}$ the group of unitary characters of Z.Let $H \subseteq G$ be a closed subgroup. Suppose that HZ is closed and that $HZ\backslash G$ carries an invariant positive measure $\mu_{HZ\backslash G}$. For a fixed $\chi \in \widehat{Z}$ we consider the Hilbert space of sections

$$\begin{aligned}
\Gamma_\chi^2(H\backslash G) = \{f\colon H\backslash G \to \mathbb{C}\colon f \text{ measurable}, (\forall z \in Z)(\forall g \in G)\; f(Hzg) = \chi(z) f(Hg);\\
\langle f, f\rangle_\chi \colon= \int_{HZ\backslash G} |f(Hg)|^2 \; d\mu_{HZ\backslash G}(HZg) < \infty\}.
\end{aligned}$$

Let $(\pi, \mathcal{H})$ be an irreducible unitary H-spherical representation of G with central character χ. Then for all $\nu \in (\mathcal{H}^{-\omega})^H$ and $v \in \mathcal{H}^\omega$ we define a continuous section by

$$\pi_{v,\nu}: H\backslash G \to \mathbb{C}, \quad Hg \mapsto \overline{\langle \nu, \pi(g).v \rangle}.$$

We say that $(\pi, \mathcal{H})$ belongs to the *relative discrete series of* $H\backslash G$, if there exists non-zero elements $\nu \in (\mathcal{H}^{-\omega})^H$ and $v \in \mathcal{H}^\omega$ such that $\pi_{v,\nu}$ belongs to $\Gamma^2_\chi(H\backslash G)$. We denote $(\mathcal{H}^{-\omega})^H_2$ the subspace of $(\mathcal{H}^{-\omega})^H$ which corresponds to the relative discrete series for $H\backslash G$. ■

In the proof of the following Proposition we adapt a nice idea of J. Faraut to our setting (cf. [Gr96, Sect. III.3]).

Proposition III.2. (Orthogonality Relations) *Let G be a Lie group with center Z. Then, if H is a closed subgroup of G such that HZ is closed and $HZ\backslash G$ carries a positive G-invariant measure, then the following assertions hold:*

(i) *If $(\pi, \mathcal{H})$ belongs to the relative discrete series of $H\backslash G$ transforming under the central character $\chi \in \widehat{Z}$ and $0 \neq \nu \in (\mathcal{H}^{-\omega})^H_2$, then all matrix coefficients $\pi_{v,\nu}$, $v \in \mathcal{H}^\omega$, belong to $\Gamma^2_\chi(H\backslash G)$ and there exists a constant $d(\pi,\nu)$ depending on the equivalence class of π and on ν such that the mapping*

$$T: \mathcal{H}^\omega \to \Gamma^2_\chi(H\backslash G), \quad v \mapsto \sqrt{d(\pi,\nu)}\pi_{v,\nu}$$

extends to a G-equivariant isometry.

(ii) *If $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$ are two inequivalent representations of the relative discrete series of $HZ\backslash G$ transforming under the same central character for Z, then for $\nu \in (\mathcal{H}^{-\omega})^H_2$ and $\eta \in (\mathcal{K}^{-\omega})^H_2$ one has*

$$\langle \pi_{v,\nu}, \sigma_{w,\eta} \rangle = \int_{HZ\backslash G} \langle \nu, \pi(g).v \rangle \overline{\langle \eta, \sigma(g).w \rangle}\; d\mu_{HZ\backslash G}(HZg) = 0$$

for all $v \in \mathcal{H}^\omega$ and $w \in \mathcal{K}^\omega$.

Proof. (i) (cf. [Gr96, Sect. III.3]) Let $D := \{v \in \mathcal{H}^\omega : \pi_{v,\nu} \in \Gamma^2_\chi(H\backslash G)\}$ and consider the unbounded operator

$$S: D \to \Gamma^2_\chi(H\backslash G), \quad v \mapsto \pi_{v,\nu}.$$

Since $\mu_{HZ\backslash G}$ is G-invariant by assumption, the same holds for D and therefore D is dense in $\mathcal{H}$ by the irreducibility of $(\pi, \mathcal{H})$. We define a positive hermitian form on D by

$$(v|w) := \langle v, w \rangle + \langle S.v, S.w \rangle_\chi \tag{3.1}$$

for $v, w \in D$. Denote by $\overline{D}$ the Hilbert completion of D with respect to $(\cdot|\cdot)$ and denote the extension of $(\cdot|\cdot)$ to its completion by the same symbol. Since $\overline{D}$ is continuously embedded into $\mathcal{H}$, there exists a bounded selfadjoint injective operator $A \in B(\mathcal{H})$ such that $\operatorname{im} A = \overline{D}$ and $(A.v|w) = \langle v, w \rangle$ for all $v \in \mathcal{H}$, $w \in \overline{D}$. Since $\langle \cdot, \cdot \rangle_\chi$ is G-invariant by the G-invariance of $\mu_{HZ\backslash G}$, it follows from (3.1) that A commutes with $\pi(G)$. Now Schur's Lemma applies and yields $A = c\,\mathrm{id}$ for some constant $c > 0$. Thus we deduce from (3.1) that

$$\langle S.v, S.w \rangle_\chi = \Big(\frac{1}{c} - 1\Big)\langle v, w \rangle$$

for all $v, w \in D$. In particular $d(\pi,\nu) := \big(\frac{1}{c} - 1\big) > 0$. Moreover S being weakly continuous, its extension to $\mathcal{H}^\omega$ coincides with $\frac{1}{\sqrt{d(\pi,\nu)}}T$, concluding the proof of (i).

(ii) Let $T_\pi: \mathcal{H} \to \Gamma^2_\chi(H\backslash G)$ and $T_\sigma: \mathcal{K} \to \Gamma^2_\chi(H\backslash G)$ be the equivariant isometric embeddings from (i). If $\operatorname{im} T_\pi \cap \operatorname{im} T_\sigma \neq \{0\}$, then

$$T_\sigma^* \circ T_\pi: \mathcal{H} \to \mathcal{K}$$

describes a non-trivial G-equivariant map. By Schur's Lemma $T_\sigma^* \circ T_\pi$ is a scalar multiple of an isometric isomorphism, contradicting the inequivalence of $(\pi, \mathcal{H})$ and $(\sigma, \mathcal{K})$. ■

Remark III.3. If $H\backslash G$ is a semisimple symmetric space, then the space $(\mathcal{H}^{-\omega})_2^H = (\mathcal{H}^{-\infty})_2$ is finite dimensional (cf. [Ba87, Th. 3.1]). Then Proposition III.2(i) says that one can find an inner product on $(\mathcal{H}^{-\omega})_2^H$ such that

$$(\mathcal{H}^{-\omega})_2^H \otimes \mathcal{H}^\omega \to \Gamma^2_\chi(H\backslash G), \quad \nu \otimes v \mapsto \sqrt{d(\pi,\nu)}\pi_{v,\nu}$$

extends to a G-equivariant isometry (with G acting trivially on the first factor $(\mathcal{H}^{-\omega})_2^H$ of the tensor product). ■

The formal dimension

If G denotes a unimodular locally compact group and $L \subseteq G$ a closed unimodular subgroup, then we denote by $\mu_{L\backslash G}$ a positive right G-invariant measure on the homogeneous space $L\backslash G$.

Definition III.4. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an H-spherical unitary highest weight representation of G and $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$. If v_λ is a highest weight vector for $(\pi_\lambda, \mathcal{H}_\lambda)$, then the *formal dimension* $d(\lambda)$ of $(\pi_\lambda, \mathcal{H}_\lambda)$ is defined by

$$\frac{1}{d(\lambda)} := \frac{1}{|\langle \nu, v_\lambda\rangle|^2} \int_{HZ\backslash G} |\langle \nu, \pi_\lambda(g).v_\lambda\rangle|^2 \, d\mu_{HZ\backslash G}(HZg).$$

Recall that $\langle \nu, v_\lambda\rangle \neq 0$ and that the definition of $d(\lambda)$ is independent of the particular choice of v_λ and $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$ (cf. Proposition II.7(ii)).

The relation between the number $d(\pi_\lambda, \nu)$ from Proposition III.2 and $d(\lambda)$ is given by $d(\lambda) = \frac{|\langle \nu, v_\lambda\rangle|^2}{\langle v_\lambda, v_\lambda\rangle} d(\pi_\lambda, \nu)$. In particular, if ν is normalized by $\frac{|\langle \nu, v_\lambda\rangle|^2}{\langle v_\lambda, v_\lambda\rangle} = 1$, then we have $d(\lambda) = d(\pi_\lambda, \nu)$. ■

Remark III.5. The particular normalization of $d(\lambda)$ as in Definition III.4 is motivated from Harish-Chandra's treatment of the "group case" (cf. [HC56]). The group case is defined by $G = G_0 \times G_0$ and $H = \Delta(G) = \{(g,g) : g \in G_0\}$ for a simply connected hermitian Lie group G_0. Then we have a natural isomorphism

$$G_0 \to H\backslash G, \quad g \mapsto H(g,1)$$

and the invariant measure $\mu_{ZH\backslash G}$ corresponds to a Haar measure $\mu_{Z(G_0)\backslash G_0}$ on $Z(G_0)\backslash G_0$.

The spherical unitary highest weight representations of G are given by $(\pi_\lambda \otimes \pi_\lambda^*, \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^*)$ with $(\pi_\lambda, \mathcal{H}_\lambda)$ a unitary highest weight representation of G_0 and $(\pi_\lambda^*, \mathcal{H}_\lambda^*)$ its dual representation. Recall that $\mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^*$ is isomorphic to the space of Hilbert-Schmidt operators $B_2(\mathcal{H}_\lambda)$ on $\mathcal{H}_\lambda$ and that the corresponding analytic vectors are of trace class, i.e., $B_2(\mathcal{H}_\lambda)^\omega \subseteq B_1(\mathcal{H}_\lambda)$ (cf. [HiKr99a, App.]). The up to scalar unique H-fixed hyperfunction vector is given by the conjugate trace:

$$\nu : B_2(\mathcal{H}_\lambda)^\omega \to \mathbb{C}, \quad A \mapsto \overline{\operatorname{tr}(A)}.$$

Further a highest weight vector for $(\pi_\lambda \otimes \pi_\lambda^*, \mathcal{H}_\lambda \widehat{\otimes} \mathcal{H}_\lambda^*)$ is given by $v_\lambda \otimes v_\lambda^*$. Then $\langle \nu, v_\lambda \otimes v_\lambda^*\rangle = \langle v_\lambda, v_\lambda\rangle$ and the expression for $d(\lambda)$ from Definition III.4 gives that

$$\frac{1}{d(\lambda)} = \frac{1}{|\langle v_\lambda, v_\lambda\rangle|^2} \int_{Z(G_0)\backslash G_0} |\langle \pi_\lambda(g).v_\lambda, v_\lambda\rangle|^2 \, d\mu_{Z(G_0)\backslash G_0}(Zg).$$

Thus we see that our definition of the formal dimension coincides in the group case with the standard one introduced by Harish-Chandra (cf. [HC56]). ■

Theorem III.6. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an unitary highest weight representations of G for which $(\pi_\lambda^K, F(\lambda))$ is $H \cap K$-spherical. Assume that $\lambda + \rho \in \mathcal{E}_\Omega$ and that $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the holomorphic discrete series of G. Then $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical, belongs to the relative discrete series of $H\backslash G$ and the formal degree $d(\lambda)$ is given by*

$$d(\lambda) = d(\lambda)^G c(\lambda + \rho),$$

where $d(\lambda)^G$ is the formal dimension of $(\pi_\lambda, \mathcal{H}_\lambda)$ relative to G.

Proof. Since $\lambda + \rho \in \mathcal{E}_\Omega$ the asumptions of Theorem II.16 are satisfied and the theorem applies. Thus $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical and if $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$ and v_λ is a highest weight vector, then we have

$$\nu = \frac{\langle \nu, v_\lambda \rangle}{\langle v_\lambda, v_\lambda \rangle c(\lambda + \rho)} \int_H \pi_\lambda(h).v_\lambda \; d\mu_H(h). \tag{3.2}$$

If we insert (3.2) in the formula for ν in the definition of the formal dimension we obtain that

$$\begin{aligned}
\frac{1}{d(\lambda)} &= \frac{1}{|\langle \nu, v_\lambda \rangle|^2} \int_{HZ\backslash G} |\langle \nu, \pi_\lambda(g).v_\lambda \rangle|^2 \; d\mu_{HZ\backslash G}(HZg) \\
&= \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \int_{HZ\backslash G} \int_H \int_H \langle \pi_\lambda(h_1).v_\lambda, \pi_\lambda(g).v_\lambda \rangle \\
&\qquad \langle \pi_\lambda(g).v_\lambda, \pi_\lambda(h_2).v_\lambda \rangle \; d\mu_H(h_1) \; d\mu_H(h_2) \; d\mu_{HZ\backslash G}(HZg) \\
&= \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \int_{HZ\backslash G} \int_H \int_H \langle \pi_\lambda(h_2 h_1).v_\lambda, \pi_\lambda(g).v_\lambda \rangle \\
&\qquad \langle \pi_\lambda(h_2^{-1} g).v_\lambda, v_\lambda \rangle \; d\mu_H(h_1) \; d\mu_H(h_2) \; d\mu_{HZ\backslash G}(HZg) \\
&= \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \int_{HZ\backslash G} \int_H \int_H \langle \pi_\lambda(h_1).v_\lambda, \pi_\lambda(h_2 g).v_\lambda \rangle \\
&\qquad \langle \pi_\lambda(h_2 g).v_\lambda, v_\lambda \rangle \; d\mu_H(h_1) \; d\mu_H(h_2) \; d\mu_{HZ\backslash G}(HZg) \\
&= \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \int_H \int_{HZ\backslash G} \int_H \langle \pi_\lambda(h_1).v_\lambda, \pi_\lambda(h_2 g).v_\lambda \rangle \\
&\qquad \langle \pi_\lambda(h_2 g).v_\lambda, v_\lambda \rangle \; d\mu_H(h_2) \; d\mu_{HZ\backslash G}(HZg) \; d\mu_H(h_1) \\
&= \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \int_H \int_{Z\backslash G} \langle \pi_\lambda(h_1).v_\lambda, \pi_\lambda(g).v_\lambda \rangle \; \langle \pi_\lambda(g).v_\lambda, v_\lambda \rangle \; d\mu_{Z\backslash G}(Zg) \; d\mu_H(h_1).
\end{aligned}$$

Thus if we apply the Harish-Chandra-Godement Orthogonality Relations for $L^2(Z\backslash G)$ and once more (3.2) we obtain that

$$\begin{aligned}
\frac{1}{d(\lambda)} &= \frac{1}{d(\lambda)^G} \cdot \frac{1}{\langle v_\lambda, v_\lambda \rangle^2 c(\lambda+\rho)^2} \langle v_\lambda, v_\lambda \rangle \int_H \langle \pi_\lambda(h).v_\lambda, v_\lambda \rangle \; d\mu_H(h) \\
&= \frac{1}{d(\lambda)^G} \cdot \frac{1}{\langle v_\lambda, v_\lambda \rangle c(\lambda+\rho)^2} c(\lambda+\rho) \langle v_\lambda, v_\lambda \rangle = \frac{1}{d(\lambda)^G c(\lambda+\rho)},
\end{aligned}$$

as was to be shown. ■

IV. Analytic continuation in λ

In this section we prove the analytic continuation of the formula for the formal dimension $d(\lambda)$ from Theorem III.6. The proof is quite technical and we need some preparation on algebraic and analytic level.

Algebraic preliminaries

In this subsection we collect some facts concerning the fine structure theory of compactly causal symmetric Lie algebras. The results are mainly due to Ólafsson (cf. [Ól91]).

Lemma IV.1. *Let $(\mathfrak{g}, \tau)$ be a compactly causal symmetric Lie algebra, then we can choose root vectors $E_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$, $\alpha \in \widehat{\Delta}_n$, such that the following conditions are satisfied:*

(1) $\overline{E_\alpha} = E_{-\alpha}$.

(2) $\alpha(H_\alpha) = 2$ *with* $H_\alpha = [E_\alpha, E_{-\alpha}]$.

(3) $\tau(E_\alpha) = E_{\tau\alpha}$, *where* $\tau\alpha = \tau \circ \alpha$.

Proof. Let κ denote the Cartan-Killing form on $\mathfrak{g}_\mathbb{C}$ and define a hermitian inner product on $\mathfrak{g}_\mathbb{C}$ by $\langle X, Y\rangle := -\kappa(X, \theta(\overline{Y}))$.

For each $\alpha \in \widehat{\Delta}_n^+$ let $0 \neq E_\alpha \in \mathfrak{g}_\mathbb{C}^\alpha$ be an arbitrary element of length 1. Then define $E_{-\alpha}$ by $E_{-\alpha} := \overline{E_\alpha}$. Thus (1) is satisfied. Now $\tau(E_\alpha) \subseteq \mathbb{C}E_{\tau\alpha}$ implies the existence of complex numbers c_α such that $\tau(E_\alpha) = c_\alpha E_{\tau\alpha}$. Now τ being an involutive implies $c_\alpha c_{\tau\alpha} = 1$, further τ being an isometry implies that $|c_\alpha| = 1$ and finally τ being complex linear implies that $\overline{c_\alpha} = c_{-\alpha}$ for all $\alpha \in \widehat{\Delta}_n$. Thus $c_{\tau\alpha} = \overline{c_\alpha} = c_{-\alpha}$. For each complex number $z = e^{i\varphi}$, $\varphi \in [0, 2\pi[$, of modulus 1 we define $z^{\frac{1}{2}} = e^{i\frac{\varphi}{2}}$. Thus redifining E_α, $\alpha \in \widehat{\Delta}_n^+$, by $\overline{c_\alpha}^{\frac{1}{2}} E_\alpha$, leaves (1) untouched and in addition satisfies (3).

Since $\mathfrak{g}_\mathbb{C}^\alpha \subseteq \mathfrak{p}_\mathbb{C}$ for all $\alpha \in \widehat{\Delta}_n^+$, we have $\alpha([E_\alpha, E_{-\alpha}]) > 0$, and so by rescaling E_α with an appropriate positive number we may in addition assume that (2) holds. This proves the lemma.∎

Let $\widehat{\Gamma} = \{\widehat{\gamma}_1, \ldots, \widehat{\gamma}_r\}$ be a maximal system of strongly orthogonal, i.e., $\widehat{\gamma}_j \pm \widehat{\gamma}_i$ is never a root and $\widehat{\Gamma} \subseteq \widehat{\Delta}_n^+$ has maximal many elements with respect to this property. In view of [HiÓl96, Lemma 4.1.7] or [Ól91, Sect. 3], we may choose $\widehat{\Gamma}$ invariant under $-\tau$.

For each $1 \leq j \leq r$ we set $\widehat{E}_j := E_{\widehat{\gamma}_j}$, $\widehat{E}_{-j} := E_{-\widehat{\gamma}_j}$ and $\widehat{X}_j := i(\widehat{E}_j - \widehat{E}_{-j})$. According to [HC56, Cor. to Lemma 8], the space

$$\mathfrak{e} := \bigoplus_{j=1}^r \mathbb{R}\widehat{X}_j = \bigoplus_{j=1}^r \mathbb{R}i(\widehat{E}_j - \widehat{E}_{-j})$$

is maximal abelian in $\mathfrak{p}$. Note that $\mathfrak{e}$ is τ-invariant by the special choice of the non-compact root vectors (cf. Lemma IV.5(3)) and the $-\tau$-invariance of $\widehat{\Gamma}$.

We consider the *Cayley transform*

$$C = e^{i\frac{\pi}{4}\operatorname{ad}(\sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j})}$$

which is an automorphism of $\mathfrak{g}_\mathbb{C}$. Finally we set $\widehat{H}_j := H_{\widehat{\gamma}_j}$ for all $1 \leq j \leq r$.

Lemma IV.2. *The Cayley transform C has the following properties:*

(i) *For all $1 \leq j \leq r$ one has $C(\widehat{X}_j) = \widehat{H}_j$ and $C(\widehat{H}_j) = -\widehat{X}_j$.*

(ii) *We have $i\frac{\pi}{4}(\sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j}) \in i\mathfrak{h}_{\mathfrak{p}}$. In particular, one has*

 (a) $\tau \circ C = C \circ \tau$,

 (b) $\theta \circ C = C^{-1} \circ \theta$,

(iii) *The Cayley transform yields an isomorphism $C: \mathfrak{e} \to C(\mathfrak{e})$ with $C(\mathfrak{e}) \subseteq i\mathfrak{t}$ a τ-invariant subspace.*

Proof. (i) This follows from $\mathfrak{sl}(2,\mathbb{R})$-reduction (cf. [HC56, p. 584], [HiÓl96, Lemma A.3.2(3)]).
(ii) It follows from $\mathfrak{g}_{\mathbb{C}}^{\widehat{\alpha}} \subseteq \mathfrak{p}_{\mathbb{C}}$, for all $\widehat{\alpha} \in \widehat{\Delta}_n$ and Lemma IV.1(1) that $i\frac{\pi}{4}(\sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j}) \in i\mathfrak{p}$. Further Lemma IV.1(3) and the $-\tau$-invariance of $\widehat{\Gamma}$ imply

$$\tau\Big(\sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j}\Big) = \sum_{j=1}^r \tau(\widehat{E}_j) + \tau(\widehat{E}_{-j}) = \sum_{j=1}^r E_{\tau\widehat{\gamma_j}} + E_{-\tau\widehat{\gamma_j}} = \sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j}.$$

Thus $i\frac{\pi}{4}(\sum_{j=1}^r \widehat{E}_j + \widehat{E}_{-j}) \in i\mathfrak{h}_{\mathfrak{p}}$. This proves (i).
(iii) This follows from (i) and (ii)(a). ■

Recall that $\mathfrak{e}$ is τ-invariant and write $\mathfrak{b} = \mathfrak{e} \cap \mathfrak{q}$ for the set of $-\tau$-fixed points.

Lemma IV.3. *Let $\mathfrak{c} := C(\mathfrak{b})$. Then $\mathfrak{c} \subseteq \mathfrak{a}$ and the Cayley transform yields an isomorphism $C: \mathfrak{b} \to \mathfrak{c}$.*

Proof. Since $C(\mathfrak{b}) \subseteq i\mathfrak{t}$ by Lemma IV.2(i), the fact that $\mathfrak{b} \subseteq \mathfrak{q}$ and that C commutes with τ (cf. Lemma IV.2(ii)) imply that $C(\mathfrak{b}) \subseteq i(\mathfrak{t} \cap \mathfrak{q})$. But $i(\mathfrak{t} \cap \mathfrak{q}) = \mathfrak{a}$ by the definition of $\mathfrak{a}$, proving the lemma. ■

Recall that $\mathfrak{b}$ is maximal abelian subspace of $\mathfrak{q} \cap \mathfrak{p}$ (this follows for instance from the c-dual version of Lemma 4.1.9 in [HiÓl96]) and denote by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{b})$ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{b}$. Recall that Σ is an abstract root system (cf. [Sch84, Sect. 7.2]). We write

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{b}) \oplus \bigoplus_{\varphi \in \Sigma} \mathfrak{g}^{\varphi}$$

for the corresponding root space decomposition. By Lemma IV.3, the Cayley transform induces a mapping $C^t: \mathfrak{a}^* \to \mathfrak{b}^*$, $\alpha \mapsto \alpha \circ C|_{\mathfrak{b}}$ and we set

$$\Sigma_n = C^t(\Delta_n)|_{\mathfrak{b}} \quad \text{and} \quad \Sigma_k = C^t(\Delta_k)|_{\mathfrak{b}} \backslash \{0\}.$$

Let $\Gamma = \{\frac{1}{2}(\widehat{\gamma}_j - \tau\widehat{\gamma}_j): 1 \leq j \leq r\}$ denote the restricted set of strongly orthogonal roots. Note that $\Gamma \subseteq \mathfrak{c}^*$ by Lemma IV.2(i). Thus we can write $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ for some $1 \leq s \leq r$. For each $1 \leq j \leq s$ we define $H_j \in \mathfrak{c}$ by $\gamma_j(H_j) = 2$ and $\gamma_k(H_j) = 0$ for $k \neq j$. We set $X_j := -C(H_j)$ for all $1 \leq j \leq s$. Then

$$\mathfrak{b} = \bigoplus_{j=1}^s \mathbb{R}X_j.$$

As a final algebraic tool we need explicit information on the root system Σ which is provided by Ólafsson's Theorem on double restricted root systems (cf. [Ól91, Sect. 3], [HÓØ91, Prop. 3.1]). For all $1 \leq j \leq s$ we set $\psi_j := C^t(\gamma_j)$ and note that $\psi_j(X_j) = 2$ since $C(X_j) = H_j$ (cf. Lemma IV.2(i), (ii)).

Finally we put $\Sigma^+ := C^t(\Delta^+)|_{\mathfrak{b}} \backslash \{0\}$, $\Sigma_n^+ := \Sigma_n \cap \Sigma^+$ and $\Sigma_k^+ := \Sigma_k \cap \Sigma^+$.

Theorem IV.4. (Ólafsson) *If $(\mathfrak{g},\tau)$ is compactly causal, then the following assertions concerning the double restricted root system $\Sigma=\Sigma(\mathfrak{g},\mathfrak{b})$ hold:*

(i) *The restricted root system has the following form*

$$\Sigma_k=\pm\{\frac{1}{2}(\psi_i-\psi_j):i<j\}\cup\pm\{\frac{1}{2}\psi_j\colon 1\leq j\leq s\}$$

and

$$\Sigma_n^+=\{\frac{1}{2}(\psi_i+\psi_j)\colon 1\leq i,j\leq s\}\cup\{\frac{1}{2}\psi_j\colon 1\leq j\leq s\}.$$

The second sets in Σ_k and Σ_n^+ are empty if and only if $C^4=\mathrm{id}$. If further ψ_s is chosen to be a simple root, then

$$\Sigma_k^+\subseteq\{\frac{1}{2}(\psi_i-\psi_j):i<j\}\cup\{\frac{1}{2}\psi_j\colon 1\leq j\leq s\}.$$

(ii) *All ψ_j, $1\leq j\leq s$, have the same length.*

(iii) *The conjugacy classes of the restricted root system under the Weyl group associated to Σ are given by*

(1) $\{\pm\frac{1}{2}(\psi_i\pm\psi_j)\colon 1\leq i,j\leq s, i\neq j\}$

(2) $\{\pm\psi_j\colon 1\leq j\leq s\}$

(3) $\{\pm\frac{1}{2}\psi_j\colon 1\leq j\leq s\}$

Proof. (i) Let $\widehat{\Sigma}=\widehat{\Sigma}(\mathfrak{g},\mathfrak{e})$ be the restricted root system with respect to the maximal abelian subspace $\mathfrak{e}$ and $\widehat{\Sigma}_k$, $\widehat{\Sigma}_n$ defined as above. Write $\widehat{\psi}_j\colon=C^t(\widehat{\gamma}_j)$ for all $1\leq j\leq r$. Suppose first that $\mathfrak{g}$ is simple. Then for the analogous statement for $\widehat{\Sigma}$ in stead of Σ and $\widehat{\psi}_j$ in stead of ψ_j, Harish-Chandra has proved in [HC56, Lemma 13-16] that $\widehat{\Sigma}_k,\widehat{\Sigma}_n^+$ are contained in the asserted subsets, Moore proved equality (cf. [Mo64, Th. 2]) and finally Korányi and Wolf have shown in [KoWo65, Prop. 4.4 with Remark] that the second set in $\widehat{\Sigma}_n^+$ is empty if and only if $C^4=\mathrm{id}$. Now taking restrictions to $\mathfrak{c}$ yields (i) for $\mathfrak{g}$ simple.

In the group case similar considerations lead to the same result.

(ii) This can be deduced from [Mo64, Th. 2(2)], but we propose here a much simpler proof. We use (i) and the fact that Σ is an abstract root system. As usual we write s_ψ, $\psi\in\Sigma$, for the reflection associated to ψ. Then we obtain for all $1\leq i\neq j\leq s$ that

$$\begin{aligned}s_{\frac{1}{2}(\psi_i+\psi_j)}(\psi_j)&=\psi_j-\frac{2\langle\psi_j,\frac{1}{2}(\psi_i+\psi_j)\rangle}{\langle\frac{1}{2}(\psi_i+\psi_j),\frac{1}{2}(\psi_i+\psi_j)\rangle}\frac{1}{2}(\psi_i+\psi_j)\\&=\psi_j-\frac{2\langle\psi_j,\psi_j\rangle}{\langle\psi_i,\psi_i\rangle+\langle\psi_j,\psi_j\rangle}(\psi_i+\psi_j).\end{aligned}$$

Thus it follows from (i) and $s_{\frac{1}{2}(\psi_i+\psi_j)}(\psi_j)\in\Sigma$ that $\frac{\langle\psi_j,\psi_j\rangle}{\langle\psi_i,\psi_i\rangle+\langle\psi_j,\psi_j\rangle}\in\{\frac{1}{2},\frac{1}{4}\}$. Interchanging i and j then yields $\frac{\langle\psi_j,\psi_j\rangle}{\langle\psi_i,\psi_i\rangle+\langle\psi_j,\psi_j\rangle}=\frac{1}{2}$ or equivalently that $\langle\psi_j,\psi_j\rangle=\langle\psi_i,\psi_i\rangle$. This proves (ii).

(iii) In view of (i), we have for all $1\leq i,j,k\leq r$ that

$$\begin{aligned}&s_{\frac{1}{2}(\psi_i\pm\psi_j)}(\psi_j)=\mp\psi_i,\\&s_{\frac{1}{2}(\psi_i\pm\psi_j)}\big(\frac{1}{2}(\psi_j\pm\psi_k)\big)=\frac{1}{2}(\mp\psi_i\pm\psi_k),\\&s_{\psi_i}\big(\frac{1}{2}(\psi_i\pm\psi_j)\big)=\frac{1}{2}(-\psi_i\pm\psi_j).\end{aligned}\tag{4.1}$$

This proves (iii). ■

From now on we assume that ψ_s is a simple root. Then Theorem IV.4(i) says that

$$\Sigma_n^+ = \{\frac{1}{2}(\psi_i + \psi_j): 1 \le i, j \le s\} \cup \{\frac{1}{2}\psi_j: 1 \le j \le s\}. \tag{4.2}$$

and

$$\Sigma_k^+ = \{\frac{1}{2}(\psi_i - \psi_j): 1 \le i < j \le s\} \cup \{\frac{1}{2}\psi_j: 1 \le j \le s\}. \tag{4.3}$$

Further it follows from Theorem IV.4(i) and the first formula in (4.1) that the Weyl group $\mathcal{W}(\Sigma_k)$ of Σ_k acts on $\mathfrak{b}$ as the full permutation group of the X_j's.

We write $\mathfrak{b}^+ = \{X \in \mathfrak{b}: (\forall\varphi \in \Sigma^+)\ \varphi(X) \ge 0\}$ for the Weyl chamber corresponding to Σ^+. By (4.2) and (4.3) we then have

$$\mathfrak{b}^+ = \{\sum_{j=1}^s x_j X_j: 0 \le x_s \le \ldots \le x_1\}.$$

Further let $\mathfrak{a}^+ := \{X \in \mathfrak{a} : (\forall\alpha \in \Delta^+)\ \alpha(X) \ge 0\}$ and $\mathfrak{c}^+ := \mathfrak{a}^+ \cap \mathfrak{c}$. Note that $C(\mathfrak{b}^+) = \mathfrak{c}^+$ by the construction of Σ^+.

Lemma IV.5. *The following equality holds*

$$C_{\min}^\star \cap (-\check{C}_k^\star) = (\mathfrak{c}^+)^\star \cap (-\check{C}_k^\star),$$

where the stars $\star$ are all taken in $\mathfrak{a}^*$.

Proof. First recall some basic rules in dealing with convex cones (cf. [Ne99b, Ch. V]). If W is a closed convex cone in an euclidean space V, then $(W^\star)^\star = W$. Further for two closed convex cones $W_1, W_2 \subseteq V$ we have $(W_1 \cap W_2)^\star = \overline{W_1^\star + W_2^\star}$.

Let now the convex cone on the left hand side be denoted by W_1, the other one by W_2. Let $p: \mathfrak{a} \to \mathfrak{c}$ be the orthogonal projection with respect to the Cartan-Killing form. We claim that $p(W_1^\star) = p(W_2^\star)$. Assume first that no half roots in Σ occur. Then from the Cayley-transform analogs of (4.2) and (4.3) it follows that

$$p(W_1^\star) = p(\overline{C_{\min} - \check{C}_k}) = \bigoplus_{j=1}^s \mathbb{R}^+ H_j + \bigoplus_{j=1}^{s-1} \mathbb{R}^+(H_{j+1} - H_j),$$

and

$$p(W_2^\star) = p(\overline{\mathfrak{c}^+ - \check{C}_k}) = \Big((\bigoplus_{j=1}^s \mathbb{R}^+ H_j) \cap \{\sum_{j=1}^s x_j H_j : x_s \le \ldots \le x_1\} \Big) + \bigoplus_{j=1}^{s-1} \mathbb{R}^+(H_{j+1} - H_j).$$

From these two equalities the claim follows in the case of no half roots in Σ. The general case is easily deduced from this.

Let $r: \mathfrak{a}^* \to \mathfrak{c}^*$, $r(\lambda) := \lambda|_\mathfrak{c}$ be the restriction map and note that r is the dual map of the inclusion mapping $\mathfrak{c} \to \mathfrak{a}$. Since both W_1 and W_2 are closed, we have $(W_{1,2}^\star)^\star = W_{1,2}$, and so

$$W_{1,2}|_\mathfrak{c} = r(W_{1,2}) = (p(W_{1,2}^\star))^\star.$$

Hence our claim implies that $W_1|_\mathfrak{c} = W_2|_\mathfrak{c}$. Thus $W_1 \subseteq W_2$ by the definition of W_1 and W_2.

For the converse inclusion we first note that an element $\lambda \in -\check{C}_k^\star$ belongs to W_1 if and only if $\lambda(\check{\beta}) \ge 0$, where β is the highest root (this becomes clear from our construction of the positive systems). Recall that $\widehat{\Gamma}$ can be constructed inductively starting with the highest root (cf. [HC56, p. 108]). Thus $\beta = \gamma_1 in \Gamma$. Hence $W_1 = (\gamma_1)^\star \cap -\check{C}_k^\star$, and so $W_2 \subseteq W_1$ since $(\mathfrak{c}^+)^\star \subseteq (\gamma_1)^\star$. ■

Analytic preliminaries

Recall the definition of $\mathfrak{b}^+$ and set $B^+ := \exp(\mathfrak{b}^+)$.

Lemma IV.6. (Flensted-Jensen) *Let $L = Z_{H\cap K}(\mathfrak{b})$. Then for the homegeneous space $HZ\backslash G$ the following assertions hold:*

(i) *The subgroups HZ and LZ of G are closed and $Z\backslash LZ$ is compact.*

(ii) *The mapping*

$$\Phi: B^+ \times LZ\backslash K \to HZ\backslash G, \quad (b, LZk) \mapsto LZbk$$

is a diffeomorphism onto its open image. The image is dense with complement of Haar measure zero.

(iii) *Up to normalization of measures we have for all $f \in L^1(HZ\backslash G)$ the following integration formula*

$$\int_{HZ\backslash G} f(HZg)\ d\mu_{HZ\backslash G}(HZg) = \int_{Z\backslash K}\int_{\mathfrak{b}^+} f(HZ\exp(X)k)\ J(X)\ dX\ d\mu_{Z\backslash K}(Zk)$$

with

$$J(X) = \prod_{\varphi\in\Sigma^+} \cosh(\varphi(X))^{m_\varphi^+} \sinh(\varphi(X))^{m_\varphi^-},$$

where $m_\varphi^\pm := \dim(\{X \in \mathfrak{g}^\varphi : \theta\tau(X) = \pm X\})$.

Proof. (i) The closedness of HZ and LZ follows from the closedness of $\mathrm{Ad}(H)$ and $Z_{\mathrm{Ad}(H)}(\mathfrak{b})$ in the adjoint group $\mathrm{Ad}(G)$. Finally $Z\backslash LZ$ is a closed subgroup of the compact group $Z\backslash Z(H\cap K)$ and hence compact.
(ii) [Sch84, Prop. 7.1.3].
(iii) It follows from [FJ80, Th. 2.6] or [Sch84, Lemma 8.1.2] that

$$J(X) := \det\big(d\Phi(X, LZk)\big) = \prod_{\varphi\in\Sigma^+} \cosh(\varphi(X))^{m_\varphi^+} \sinh(\varphi(X))^{m_\varphi^-}$$

for all $X \in \mathfrak{b}^+$ and $k \in K$. Thus it follows from (ii) that

$$\int_{HZ\backslash G} f(HZg)\ d\mu_{HZ\backslash G}(HZg) = \int_{LZ\backslash K}\int_{\mathfrak{b}^+} f(HZ\exp(X)k)\ J(X)\ dX\ d\mu_{LZ\backslash K}(Zk)$$

holds for all $f \in L^1(HZ\backslash G)$. In view of (i), we may replace the integration over $LZ\backslash K$ by an $Z\backslash K$-integral, proving (iii). ∎

Lemma IV.7. *Realize G as a submanifold of $\widetilde{M} \times P^+$ as in* Proposition II.10(ii). *Then for $b = \exp_G(\sum_{j=1}^s x_j X_j) \in B$ and*

$$\mu(b) := \exp_{\widetilde{K_\mathbb{C}}}\Big(\sum_{j=1}^s \frac{1}{2}\log\cosh(2x_j)H_j\Big) \in A \subseteq \widetilde{K_\mathbb{C}}$$

the following assertions hold:

(i) *We have $b \in \{[h, \mu(b)] : h \in \widetilde{H_\mathbb{C}}\} \times P^+$.*

(ii) *If $X \in \mathfrak{b}^+$, then $\log\mu(\exp_G(X)) \in \mathfrak{c}^+$.*

Proof. (i) This follows directly from [HiÓl96, p. 210-211].
(ii) Recall that $X = \sum_{j=1}^s x_j X_j \in \mathfrak{b}^+$ if and only if $0 \le x_s \le \ldots \le x_1$. Now the assertion follows from (i) and the monotonicity of the mapping $\mathbb{R}^+ \to \mathbb{R},\ x \mapsto \log\cosh(x)$. ∎

Proof of the analytic continuation

Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an H-spherical unitary highest weight representation of G. Further let $\nu \in (\mathcal{H}_\lambda^{-\omega})^H$ an H-fixed hyperfunction vector and $\nu_0 = \nu|_{F(\lambda)} \in F(\lambda)^{H \cap K}$. We normalize ν by setting $\|\nu_0\| = 1$ and then v_λ by $|\langle \nu, v_\lambda \rangle| = 1$. Then we have

$$d(\lambda) = I(\lambda)^{-1} \quad \text{with} \quad I(\lambda) := \int_{HZ\backslash G} |\langle \nu, \pi_\lambda(g).v_\lambda \rangle|^2 \; d\mu_{HZ\backslash G}(HZg).$$

Definition IV.8. On the non-compactly Riemannian symmetric space $K(0)\backslash G(0)$ we define the *spherical function with parameter* $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by

$$\varphi_\lambda^0(g) = \int_{K(0)} a_H(gk)^{\lambda - \rho_k} \; d\mu_{K(0)}(k),$$

for all $g \in G(0)$. ■

Remark IV.9. Note that if $\lambda \in \mathfrak{a}^*$ is the highest weight of an $H \cap K$-spherical representation $(\pi_\lambda^K, F(\lambda))$ of $\widetilde{K_{\mathbb{C}}}$, then $\varphi^0_{\lambda+\rho_k}$ extends to a holomorphic function on $\widetilde{K_{\mathbb{C}}}$ and we have

$$\varphi^0_{\lambda+\rho_k}(k) = \langle \pi_\lambda^K(k).\nu_0, \nu_0 \rangle \tag{4.4}$$

for all $k \in \widetilde{K_{\mathbb{C}}}$ (cf. [Hel84, Ch. V, Th. 4.3]). ■

Proposition IV.10. *With the notation of* Lemma IV.7 *we have*

$$I(\lambda) = \frac{1}{\dim F(\lambda)} \int_{\mathfrak{b}^+} \varphi^0_{\lambda+\rho_k}(\mu(\exp_G(X))^2) \; J(X) \; dX$$

where $J(X)$ *is given as in* Proposition IV.6(iii).

Proof. (cf. [HC56, p. 599], [Gr96, Prop. 10]) In the sequel we identify $\mathfrak{b}$ with B via the exponential mapping and for $b = \exp_G(X) \in B^+$ we set $J(b) := J(X)$. Then by Lemma IV.6(iii) we have

$$\begin{aligned} I(\lambda) &= \int_{HZ\backslash G} |\langle \nu, \pi_\lambda(g).v_\lambda \rangle|^2 \; d\mu_{HZ\backslash G}(HZg) \\ &= \int_{Z\backslash K} \int_{B^+} |\langle \nu, \pi_\lambda(bk).v_\lambda \rangle|^2 \; J(b) \; d\mu_B(b) \; d\mu_{Z\backslash K}(k). \end{aligned} \tag{4.5}$$

In view of Lemma II.10(ii), we can write each element in $b \in B^+$ as $([h_{\mathbb{C}}(b), \mu(b)], p_+(b)) \in \widetilde{M} \times P^+$ with $\mu(b) \in \widetilde{K_{\mathbb{C}}}$. Now the same consideration as in the proof of Step 1 of Theorem II.16 yields for all $b \in B^+$ and $k \in K$ that

$$\begin{aligned} \langle \nu, \pi_\lambda(bk).v_\lambda \rangle &= \langle \nu, \pi_\lambda\big(([h_{\mathbb{C}}(b), \mu(b)], p_+(b))k\big).v_\lambda \rangle \\ &= \langle \nu, \pi_\lambda([\mathbf{1}, \mu(b)k], k^{-1}p_+(b)k).v_\lambda \rangle = \langle \nu, \pi_\lambda(\mu(b)k).v_\lambda \rangle \\ &= \langle \nu_0, \pi_\lambda^K(\mu(b)k).v_\lambda \rangle. \end{aligned}$$

If we insert this expression for the matrix coefficient in (4.5), use Schur's Orthogonality Relations for $(\pi_\lambda^K, F(\lambda))$ and the relation $\pi_\lambda^K(\mu(b))^* = \pi_\lambda^K(\mu(b))$ (cf. Lemma IV.7), we arrive at

$$\begin{aligned} I(\lambda) &= \int_{B^+} \int_{Z\backslash K} |\langle \nu_0, \pi_\lambda^K(\mu(b)k).v_\lambda\rangle|^2 \; J(b) \; d\mu_{Z\backslash K}(k) \; d\mu_B(b) \\ &= \frac{1}{\dim F(\lambda)} \int_{B^+} \langle \pi_\lambda^K(\mu(b)).\nu_0, \pi_\lambda^K(\mu(b)).\nu_0\rangle \; J(b) \; d\mu_B(b) \\ &= \frac{1}{\dim F(\lambda)} \int_{B^+} \langle \pi_\lambda^K(\mu(b)^2).\nu_0, \nu_0\rangle \; J(b) \; d\mu_B(b). \end{aligned}$$

Now the assertion of the proposition follows from (4.4). ∎

Lemma IV.11. *Let V be a finite dimensional real vector space, $W \subseteq V$ be an open convex cone, $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in W^\star \backslash \{0\}$ and $p_1, \ldots, p_n, q_1 \ldots, q_m \in \mathbb{N}_0$. For every $\lambda \in V^*$ we define the integral*

$$H(\lambda) := \int_W e^{\lambda(x)} \prod_{j=1}^n (\sinh \alpha_j(x))^{p_j} \prod_{\mathrm{j}=1}^m \big(\cosh \beta_j(x)\big)^{q_j} \; d\mu_V(x).$$

Then $H(\lambda)$ converges if and only if $\lambda + \sum_{j=1}^n p_j \alpha_j + \sum_{j=1}^m q_j \beta_j \in -\operatorname{int} W^\star$.

Proof. If $q_1 = \ldots = q_m = 0$, then this is Lemma IV.6 in [Kr98]. The general case is easily obtained from this. ∎

The following characterization of the relative discrete series by the parameter λ is due to Hilgert, Ólafsson and Ørsted and was obtained in two steps (cf. [ÓØ91, Th. 5.2], [HÓØ91, Th. 3.3]). We present an essentially modified proof here, but we point out that it is not our objective to give new proofs of well-known facts. In the course of our arguments we obtain an important new estimate which is crucial for the analytic continuation of $I(\lambda)$.

Theorem IV.12. (Hilgert-Ólafsson-Ørsted) *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an unitary highest weight representation of G with $(\pi_\lambda^K, F(\lambda))$ being $H \cap K$-spherical. Then $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the relative discrete series of $H\backslash G$ if and only if the condition*

$$\text{(RDS)} \qquad (\forall \alpha \in \Delta_n^+) \quad \langle \lambda + \rho, \alpha \rangle < 0$$

is satisfied.

Proof. Recall the definition of $\mathfrak{c}^+$, $\mathfrak{a}^+$ and the relation $C(\mathfrak{b}^+) = \mathfrak{c}^+$. Set $A^+ := \exp_{G^c}(\mathfrak{a}^+)$ and let $\|\cdot\|$ denote an arbitrary norm on $\mathfrak{a}$. If we write $(\mathfrak{c}^+)^\star$, then the star $\star$ is to be taken in $\mathfrak{a}^*$.

Step 1: $I(\lambda) < \infty$, if $\lambda + \rho \in -\operatorname{int}(\mathfrak{c}^+)^\star$, the interior of $(\mathfrak{c}^+)^\star$.

Here we do not assume that $\lambda \in \mathfrak{a}^*$ is dominant integral with respect to Δ_k^+, but only $\lambda \in \check{C}_k^\star$. By Harish-Chandra's estimates for spherical functions on non-compact Riemannian symmetric spaces, there exists constants $c > 0$ and $d \in \mathbb{N}$ such that

$$(4.6) \qquad (\forall \lambda \in \check{C}_k^\star)(\forall a \in A^+) \quad \varphi_\lambda^0(a) \le c a^{\lambda - \rho_k} (1 + \|\log a\|)^d$$

(cf. [Wal88, 4.5.3]). Note that $J(X) \le e^{2\rho(C(X))}$ for all $X \in \mathfrak{b}^+$ by the formula for the Jacobian in Lemma IV.6(iii). Thus it follows for all $\lambda \in \check{C}_k^\star$ and $X = \sum_{j=1}^s x_j X_j \in \mathfrak{b}^+$ from (4.6) together with Lemma IV.7 that

$$(4.7) \qquad \begin{aligned} \varphi_{\lambda+\rho_k}^0(\mu(\exp_G(X))^2) \; J(X) &\le c\mu(\exp_G(X))^{2\lambda}(1 + \|\log \mu(\exp_G(X))^2\|)^d e^{2\rho(C(X))} \\ &\le c e^{2\lambda(C(X))}(1 + 2\|C(X)\|)^d e^{2\rho(C(X))} \\ &\le c e^{2(\lambda+\rho)(C(X))}(1 + 2\|C(X)\|)^d. \end{aligned}$$

Now Proposition IV.11 shows that $I(\lambda) < \infty$ if $\lambda + \rho \in -\operatorname{int}(\mathfrak{c}^+)^\star$, proving our first step.

Step 2: $\lambda + \rho \in -\operatorname{int}(\mathfrak{c}^+)^\star$, if $I(\lambda) < \infty$.

Recall that λ is supposed to be dominant integral with respect to Δ_k^+. Thus it follows from (4.4) and the fact that the $H \cap K$-spherical vector ν_0 has a non-zero v_λ-component (cf. [Hel84, p. 537, (7)]) that there is a constant $c_\lambda > 0$ such that $c_\lambda a^\lambda \leq \varphi^0_{\lambda+\rho_k}(a)$ holds for all $a \in A^+$. Hence Lemma IV.7 implies that

$$(\forall X \in \mathfrak{b}^+) \quad \frac{c_\lambda}{2} e^{2\lambda(C(X))} J(X) \leq \varphi^0_{\lambda+\rho_k}(\mu(\exp_G(X))^2) J(X).$$

In view of Proposition IV.10 and Lemma IV.11, we obtain $\lambda + \rho \in -\operatorname{int}(\mathfrak{c}^+)^\star$ if $I(\lambda) < \infty$. This proves our second step.

Step 3: If $\lambda \in \check{C}_k^\star$, then λ satisfies (RDS) if and only if $\lambda + \rho \in -\operatorname{int}(\mathfrak{c}^+)^\star$.

Note that λ satisfies (RDS) means that $\lambda + \rho \in -\operatorname{int} C^\star_{\min}$. Now if $\lambda \in \check{C}_k^\star$, then $\lambda + \rho \in \operatorname{int} \check{C}_k^\star$. Thus Step 3 follows from Lemma IV.5.

In view of Steps 1-3, it follows that $I(\lambda)$ is finite if and only if λ satisfies the condition (RDS). The proof of the theorem will therefore be complete with

Step 4: If λ satisfies (RDS), then $(\pi_\lambda, \mathcal{H}_\lambda)$ is H-spherical.

Let $\kappa\colon G \to \widetilde{K_{\mathbb{C}}}/(\widetilde{K_{\mathbb{C}}} \cap \widetilde{H_{\mathbb{C}}})_0$ the canonical projection defined via the decomposition in Proposition II.10. Now the function

$$H\backslash G \to \mathbb{C}, \quad Hg \mapsto \langle \pi^K_\lambda(\kappa(g)).v_\lambda, \nu_0\rangle$$

generates an H-spherical module in the relative discrete series on $H\backslash G$ (cf. [ÓØ91, Th. 5.2]). This proves Step 4 and concludes the proof of the theorem. ∎

The prescription

$$W := -\operatorname{int} C^\star_{\min} \cap \check{C}_k^\star \subseteq -\operatorname{int}(\mathfrak{c}^+)^\star$$

defines a convex cone in $\mathfrak{a}^*$. We write $T_W = i\mathfrak{a}^* + W$ for the associated tube domain in $\mathfrak{a}^*_{\mathbb{C}}$. Note that $\rho_n \in i\mathfrak{z}(\mathfrak{k})^*$ by the construction of Δ_n^+ and so $-\rho_n \in W$.

Lemma IV.13. *The function $I(\lambda)$ extends naturally to a continuous function on the affine subtube $T_W - \rho$, also denoted by I, and which is holomorphic when restricted to $T_{W^0} - \rho$. If $m \in \mathbb{N}$ is sufficiently large, then $W - m\rho_n \subseteq W - \rho$ and $I|_{T_W - m\rho_n}$ is bounded.*

Proof. First we show that $W - m\rho_n \subseteq W - \rho$ for large values of $m \in \mathbb{N}$. Since $\rho_n \in \operatorname{int} C^\star_{\min}$, we have $\rho - m\rho_n \in -\operatorname{int} C^\star_{\min}$ provided $m \in \mathbb{N}$ is sufficiently large. Further $\rho_n \in i\mathfrak{z}(\mathfrak{k})^*$ shows that $\mathbb{R}.\rho_n \in \check{C}_k^\star$. Thus we have $\rho - m\rho_n \in W$ if m is chosen sufficiently large, proving our claim.

Recall the formula for $I(\lambda)$ from Proposition IV.10. Then (4.7) yields constants $c > 0$, $d \in \mathbb{N}$ such that

$$I(\lambda) \leq \frac{c}{\dim F(\lambda)} \int_{\mathfrak{c}^+} e^{2(\lambda+\rho)(X)} (1 + 2\|X\|)^d \, dX \tag{4.8}$$

holds for some norm $\|\cdot\|$ on $\mathfrak{a}$. Let $\widehat{\rho}_k$ denote the half sum of the roots in $\widehat{\Delta}_k^+$ and recall Weyl's Dimension Formula

$$\dim F(\lambda) = \frac{\prod_{\widehat{\alpha} \in \widehat{\Delta}_k^+} \langle \lambda + \widehat{\rho}_k, \widehat{\alpha}\rangle}{\prod_{\widehat{\alpha} \in \widehat{\Delta}_k^+} \langle \widehat{\rho}_k, \widehat{\alpha}\rangle}.$$

In particular, we see that $\lambda \mapsto \frac{1}{\dim F(\lambda)}$ extends to a holomorphic map on T_W and $T_W - \rho$ which is bounded when restricted to $T_W - m\rho_n$ for all $m \in \mathbb{N}_0$. Further for each fixed $b \in B^+$ the mapping

$$\mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}, \quad \lambda \mapsto \varphi^0_{\lambda+\rho_k}(\mu(b)^2)$$

is holomorphic. Now (4.8) together with Proposition IV.10 imply that $I(\lambda)$ extends to a continuous function on $T_W - \rho$ which is holomorphic on $T_{W^0} - \rho$ and bounded when restricted to $T_W - m\rho_n$ provided m is chosen sufficiently large. ∎

Lemma IV.14. *If $m \in \mathbb{N}$ is sufficiently large, then the function*

$$T_{W^0} - m\rho_n \to \mathbb{C}, \quad \lambda \mapsto c(\lambda + \rho)$$

is holomorphic and bounded.

Proof. In view of $\rho_n \in i\mathfrak{z}(\mathfrak{k})^*$, this is immediate from Theorem II.14. ∎

Theorem IV.15. (The formal dimension for the relative holomorphic discrete series on a compactly causal symmetric space) *Let $H\backslash G$ be a simply connected symmetric space associated to a compactly causal symmetric Lie algebra $(\mathfrak{g}, \tau)$ and $(\pi_\lambda, \mathcal{H}_\lambda)$ be an unitary highest weight representations of G for which $F(\lambda)$ is $H \cap K$-spherical. Then the following assertions hold:*

(i) *The representation $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the relative discrete series for $H\backslash G$ if and only if the condition*

(RDS) $$(\forall \alpha \in \Delta_n^+) \quad \langle \lambda + \rho, \alpha \rangle < 0$$

is satisfied.

(ii) *If $(\pi_\lambda, \mathcal{H}_\lambda)$ belongs to the relative discrete series of $H\backslash G$, then the formal dimension $d(\lambda)$ is given by*

$$d(\lambda) = d(\lambda)^G c(\lambda + \rho),$$

where $d(\lambda)^G$ is the formal dimension of $(\pi_\lambda, \mathcal{H}_\lambda)$ relative to G and c is the c-function of the non-compactly c-dual space $H^c\backslash G^c$ of $H\backslash G$ (cf. Theorem II.14*). Here the right hand side has to be understood as an analytic continuation of a product of two meromorphic functions.*

Proof. (i) Theorem IV.12.

(ii) Let $\widehat{\rho}$ denote the half sum of the elements in $\widehat{\Delta}^+$ and recall Harish-Chandra's condition for the relative discrete series on G

$$(\forall \widehat{\alpha} \in \widehat{\Delta}_n^+) \quad \langle \lambda + \widehat{\rho}, \widehat{\alpha} \rangle < 0$$

(cf. [HC56, Lemma 29]) as well as Harish-Chandra's formula for the formal dimension $d(\lambda)^G$ of the relative discrete series on G

$$d(\lambda)^G = \frac{\prod_{\widehat{\alpha} \in \widehat{\Delta}^+} \langle \lambda + \widehat{\rho}, \widehat{\alpha} \rangle}{\prod_{\widehat{\alpha} \in \widehat{\Delta}^+} \langle \widehat{\rho}, \widehat{\alpha} \rangle}$$

(cf. [HC56, Th. 4]). In particular for $m \in \mathbb{N}$ sufficiently large, the prescription $\lambda \mapsto \frac{1}{d(\lambda)^G}$ defines a bounded holomorphic function on the affine tube $T_{W^0} - m\rho_n$.

Now it follows from Lemma IV.13 and Lemma IV.14 that the function

$$f\colon T_{W^0} - m\rho_n \to \mathbb{C}, \quad \lambda \mapsto I(\lambda)c(\lambda + \rho) - \frac{1}{d(\lambda)^G}$$

is holomorphic and bounded for m sufficiently large. For such m Theorem III.6 implies that $f(\lambda) = 0$ for all $\lambda \in W^0 - m\rho_n$ which are dominant integral with respect to Δ_k^+. Thus the identity criterion of Proposition A.2 in Appendix A applies and yields $f = 0$. We conclude in particular that $I(\lambda)^{-1}$ defines a continuation of $\lambda \mapsto d(\lambda)^G c(\lambda + \rho)$ to a continuous function on $T_W - \rho$ which is holomorphic when restricted to the interior $T_{W^0} - \rho$. Since by definition $d(\lambda) = I(\lambda)^{-1}$, the assertion in (ii) follows because λ satisfies (RDS) if and only if $\lambda \in T_W - \rho$. ∎

The following result has already been obtained earlier by Faraut, Hilgert and Ólafsson in [FHÓ94, Lemma 9.2], but with a completely different type of arguments (see also Theorem II.14).

Corollary IV.16. *Suppose that* $(\mathfrak{g},\tau) = (\mathfrak{h}\oplus\mathfrak{h},\sigma)$ *is of group type (cf.* Lemma I.3(i)(2)*). Then the domain of convergence* $\mathcal{E}$ *for* c *is given by*

$$\mathcal{E} = i\mathfrak{a}^* + (-\operatorname{int} C_{\min}^\star) \cap \operatorname{int} \check{C}_k^\star$$

and there exists a constant $\gamma > 0$ *only depending on the choice of the various Haar measures such that*

$$c(\lambda) = \gamma \frac{1}{\prod_{\alpha\in\Delta^+}\langle\lambda,\alpha\rangle}$$

for $\lambda \in \mathcal{E}$.

Proof. In the following we use the notation of Remark III.5. Since $(\mathfrak{g},\tau)$ is of group type we have $d(\lambda)^G = d(\lambda)^{(G_0\times G_0)} = (d(\lambda)^{G_0})^2$, and so it follows from Theorem IV.15(ii) that $c(\lambda+\rho) = \frac{1}{d(\lambda)^{G_0}}$ holds for the analytic continuations. In view of Harish-Chandra's formula for $d(\lambda)^{G_0}$ (cf. [HC56, Th. 4]), this proves the corollary. ■

Problems . The discrete series on $H\backslash G$ are constructed by analytic methods, i.e., with generating functions (cf. [FJ80], [MaOs84], [ÓØ91]). But from the algebraic point of view there are still some interesting open problems.

(a) Using the classification sheme of unitary highest weight modules (cf. [EHW83]) together with the fine structure theory of compactly causal symmetric Lie algebras provided by Theorem IV.4 and [Ól91] one can check case by case that (RDS) implies that $N(\lambda) = L(\lambda)$. In view of Proposition II.7(ii), this gives a more algebraic proof of the fact that (RDS) implies that $(\pi_\lambda,\mathcal{H}_\lambda)$ is H-spherical whenever $(\pi_\lambda^K, F(\lambda))$ is $H\cap K$-spherical. The following questions are therefore natural: What is the algebraic impact of the condition (RDS)? Does there exists an analog of the Parthasarathy-condition (cf. [EHW83. Prop. 3.9]) for the symmetric space setting?

(b) Give a complete classification of H-spherical unitary highest weight representations. A first step in this direction might be Proposition II.7(ii) and Remark II.8. ■

V. Applications to holomorphic representation theory

In this final section we give a second application of the Averaging Theorem: We relate the spherical character of a spherical unitary highest weight representation of G to the corresponding spherical functions on the c-dual space.

Spherical functions and character theory

Definition V.1. Let $(\pi_\lambda,\mathcal{H}_\lambda)$ be an H-spherical unitary highest weight representation of G. If $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$ and v_λ is an highest weight vector, then we define the *spherical character* Θ_λ of $(\pi_\lambda,\mathcal{H}_\lambda)$ by

$$\Theta_\lambda\colon S_{\max}^0 \to \mathbb{C}, \quad s \mapsto \frac{\langle v_\lambda, v_\lambda\rangle}{|\langle \nu, v_\lambda\rangle|^2}\langle\pi_\lambda(s).\nu,\nu\rangle.$$

Note that Θ_λ is an H-biinvariant holomorphic function on $S_{\max}^0$ (cf. [KNÓ97, Lemma V.6]). ■

Remark V.2. The particular normalization of Θ_λ has two reasons. First that it coincides in the group case (cf. Remark III.5) with the standard definition, and second because it has the best analytic properties for the assignments $\lambda \mapsto \Theta_\lambda(s)$, $s \in S_{\max}^0$ (as less poles as possible). ■

Definition V.3. (Spherical Functions) Recall the definition of the domain $\mathcal{E}_\Omega \subseteq \mathfrak{a}_{\mathbb{C}}^*$ (cf. Definition II.12). If $\lambda \in \mathcal{E}_\Omega$, then the *spherical function with parameter* λ is defined by

$$\varphi_\lambda\colon S_{\max}^0 \cap HAN \to \mathbb{C}, \quad s \mapsto \int_H a_H(sh)^{\lambda-\rho}\, d\mu_H(h)$$

(cf. [FHÓ94] or [KNÓ98]). Recall that the defining integrals converge absolutely if and only if $\lambda \in \mathcal{E}_\Omega$ (cf. [FHÓ94, Th. 6.3]). ■

Theorem V.4. *Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be an H-spherical unitary highest weight representation of G such that $\lambda + \rho \in \mathcal{E}_\Omega$ holds. Then the spherical character Θ_λ of $(\pi_\lambda, \mathcal{H}_\lambda)$ and the spherical function $\varphi_{\lambda+\rho}$ are related by*

$$(\forall s \in S_{\max}^0 \cap HAN) \quad \Theta_\lambda(s) = \frac{1}{c(\lambda+\rho)}\varphi_{\lambda+\rho}(s).$$

In particular, $\varphi_{\lambda+\rho}$ extends naturally to a H-biinvariant holomorphic function on $S_{\max}^0$.

Proof. Since $\lambda + \rho \in \mathcal{E}_\Omega$, the assumption of Theorem II.16 is satisfied and we can rewrite $0 \neq \nu \in (\mathcal{H}_\lambda^{-\omega})^H$ as

$$\nu = \frac{\langle \nu, v_\lambda\rangle}{\langle v_\lambda, v_\lambda\rangle c(\lambda+\rho)} \int_H \pi_\lambda(h).v_\lambda\, d\mu_H(h).$$

Thus if we replace the first ν in the definition of Θ_λ by this expression, we get for all $s \in S_{\max}^0 \cap HAN$ that

$$\begin{aligned}
\Theta_\lambda(s) &= \frac{\langle v_\lambda, v_\lambda\rangle}{|\langle \nu, v_\lambda\rangle|^2}\langle \pi_\lambda(s).\nu, \nu\rangle \\
&= \frac{1}{c(\lambda+\rho)} \cdot \frac{1}{\langle v_\lambda, \nu\rangle} \int_H \langle \pi_\lambda(sh).v_\lambda, \nu\rangle\, d\mu_H(h) \\
&= \frac{1}{c(\lambda+\rho)} \cdot \frac{1}{\langle v_\lambda, \nu\rangle} \int_H \langle \pi_\lambda(h_H(sh)a_H(sh)n_H(sh)).v_\lambda, \nu\rangle\, d\mu_H(h) \\
&= \frac{1}{c(\lambda+\rho)} \cdot \frac{1}{\langle v_\lambda, \nu\rangle} \int_H \langle \pi_\lambda(a_H(sh)).v_\lambda, \nu\rangle\, d\mu_H(h) \\
&= \frac{1}{c(\lambda+\rho)} \int_H a_H(sh)^\lambda d\mu_H(h) \\
&= \frac{1}{c(\lambda+\rho)}\varphi_{\lambda+\rho}(s),
\end{aligned}$$

as was to be shown. ■

Remark V.5. (a) We remark here that the relation in Theorem V.4 was long time searched by G. Ólafsson (cf. [Ól98, Open Problem 7(1)]). For further interesting problems related to this subject we refer to [Fa98] and [Ól98].
(b) The analytic continuation of the relation in Theorem V.4 has been obtained in [HiKr98]. It has far reaching consequences for the theory of G-invariant Hilbert spaces of holomorphic functions on G-invariant subdomains of the Stein manifold $\Xi_{\max}^0 = G \times_H iW_{\max}^0$. In particular, it implies the *Plancherel Theorem* for these class of Hilbert spaces (cf. [HiKr98]). For further information related to this subject we refer to [HiKr99b], [KNÓ97], [Kr98,99b] and [Ne99a]. ■

Appendix

A. An identity criterion for bounded analytic functions on tubes

Lemma A.1. *Let $\Pi^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane and $H^\infty := \{f \in \mathrm{Hol}(\Pi^+) : \|f\|_\infty < \infty\}$ the Banach space of bounded holomorphic functions on it. Let $\alpha > 0$ and $N = \{n\alpha i : n \in \mathbb{N}\}$. Then the following identity assertion for elements f of $H^\infty(\Pi^+)$ holds: If $f|_N = 0$, then $f = 0$.*

Proof. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ and $H^\infty(D) = \{f \in \mathrm{Hol}(D) : \|f\|_\infty < \infty\}$. Let $f \in H^\infty(D)$ and $\{\beta_n : n \in \mathbb{N}\}$ be subset of zeros of f. Then it follows from [Ru70, Th. 15.23] that

$$(A.1) \qquad f = 0 \quad \text{if} \quad \sum_{n=1}^{\infty}(1 - |\beta_n|) = \infty.$$

We consider the Cayley transform

$$c : \Pi^+ \to D, \quad z \mapsto \frac{z-i}{z+i},$$

which is an biholomorphic isomorphism, defining an isomorphism of Banach spaces

$$c_* : H^\infty(D) \to H^\infty(\Pi^+), \quad f \mapsto \widetilde{f} = f \circ c.$$

Let $\alpha_n := n\alpha i$. Then we have

$$\beta_n := c(\alpha_n) = \frac{n\alpha i - i}{n\alpha i + i} = \frac{n\alpha - 1}{n\alpha + 1}.$$

Let $N_0 \in \mathbb{N}$ be such that $n\alpha - 1 > 0$ for all $n \geq N_0$. Then

$$(A.2) \qquad \sum_{n=1}^{\infty}(1 - |\beta_n|) \geq \sum_{n=N_0}^{\infty}\left(1 - \frac{n\alpha - 1}{n\alpha + 1}\right) = \sum_{n=N_0}^{\infty}\frac{2}{n\alpha + 1} = \infty.$$

Thus if $\widetilde{f} \in H^\infty(\Pi^+)$ vanishes on all α_n, $n \in \mathbb{N}$, then $f(\beta_n) = 0$ for all $n \in \mathbb{N}$ and so $f = 0$ by (A.1) and (A.2). Therefore $\widetilde{f} = c_*(f) = 0$, proving the lemma. ∎

Proposition A.2. *Let $\emptyset \neq W \subseteq \mathbb{R}^n$ be an open convex cone, $T_W := \mathbb{R}^n + iW$ the associated tube domain in $\mathbb{C}^n$ and $H^\infty(T_W) = \{f \in \mathrm{Hol}(T_W) : \|f\|_\infty < \infty\}$ the space of bounded holomorphic functions on T_W. Let $\Gamma \subseteq \mathbb{R}^n$ be a lattice. Then the following identity assertion holds:*

$$(\forall f \in H^\infty(T_W)) \quad f|_{i(\Gamma \cap W)} = 0 \ \Rightarrow \ f = 0.$$

Proof. We prove the assertion by induction on the dimension $n \in \mathbb{N}$.

If $n = 1$, then $\Gamma = \mathbb{Z}\alpha$ for some $\alpha > 0$ and $W = \mathbb{R}, \mathbb{R}^+$ or $\mathbb{R}^-$. If $W = \mathbb{R}$, then the assertion follows from Liouville's Theorem. In the two remaining cases the assertion follows from Lemma A.1.

Suppose now the assertion is true for all all dimensions less or equal to $n-1$, $n \geq 2$. Let $f \in H^\infty(\mathbb{R}^n + iW)$ be an element vanishing on $i(\Gamma \cap W)$. We have to show that $f=0$. Since W is open, we find a basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ which is contained in $\Gamma \cap W$. By the Identity Theorem for analytic functions, it suffices to prove the assertion for $\Gamma = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ and $W = \sum_{j=1}^n \mathbb{R}^+ e_j$. Let $\Gamma_{n-1} = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_{n-1}$ and $W_{n-1} = \sum_{j=1}^{n-1} \mathbb{R}^+ e_j$. Write the variables $z \in \mathbb{C}^n$ as tuples $z = (z', z_n)$ with $z' = (z_1, \ldots, z_{n-1})$. By induction we obtain that $f(z) = f(z', z_n)$ does not depend on the z'-variable. Thus $f(z) = F(z_n)$ for some $F \in H^\infty(\Pi^+)$ with $F|_{\mathbb{N}i} = 0$. Thus by the induction hypothesis $F = 0$, and hence $f = 0$ establishing the induction step. ■

B. A lemma on spherical highest weight modules

Throughout this subsection $(\mathfrak{g}, \tau)$ denotes a simple hermitian symmetric Lie algebra. Further we use the notation from Section I-II.

Lemma B.1. *Suppose that $(\mathfrak{g}, \tau)$ is a simple hermitian symmetric Lie algebra and (G, τ) an associated simply connected Lie group. Set $H = G^\tau$ and assume that there exist a non-trivial H-spherical unitary highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G. Then the symmetric Lie algebra $(\mathfrak{g}, \tau)$ has to be compactly causal.*

Proof. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for a τ-invariant Cartan decomposition of $\mathfrak{g}$ and let K denote the analytic subgroup of G corresponding to $\mathfrak{k}$.

By assumption we have $(\mathcal{H}_\lambda^{-\omega})^H \neq \{0\}$. In particular we can conclude that the module $L(\lambda)$ of K-finite vectors of $(\pi_\lambda, \mathcal{H}_\lambda)$ admits nontrivial $H \cap K$-fixed vectors. Recall that $L(\lambda)$ is the unique irreducible quotient of the generalized Verma module

$$N(\lambda) = \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+)} F(\lambda).$$

In particular, there exists an element $0 \neq v_0 \in N(\lambda)^{H \cap K}$.

Recall that $N(\lambda)$ is $\mathfrak{k}_\mathbb{C}$-isomorphic to $\mathcal{S}(\mathfrak{p}^-) \otimes F(\lambda)$, where the $\mathfrak{k}_\mathbb{C}$-action on $\mathcal{S}(\mathfrak{p}^-) \otimes F(\lambda)$ is defined by

$$X.(p \otimes v) := [X, p] \otimes v + p \otimes X.v \tag{B.1}$$

for $X \in \mathfrak{k}_\mathbb{C}$, $p \in \mathcal{S}(\mathfrak{p}^-)$ and $v \in F(\lambda)$ (cf. [EHW83]).

In order to show that $(\mathfrak{g}, \tau)$ is compactly causal, we have to prove $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{q}$. Assume the contrary, i.e., $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{h}$. Recall the definition of the element $Z_0 \in \mathfrak{z}(\mathfrak{k})$ from Section I and set $X_0 := -iZ_0 \in i\mathfrak{z}(\mathfrak{k})$. Then the spectrum of X_0, considered as an operator on the symmetric algebra $\mathcal{S}(\mathfrak{p}^-)$, is $-\mathbb{N}_0$, and we obtain a natural grading by homogeneous elements: $\mathcal{S}(\mathfrak{p}^-) = \bigoplus_{n=0}^\infty \mathcal{S}(\mathfrak{p}^-)^{-n}$. Then $N(\lambda) = \bigoplus_{n=0}^\infty \mathcal{S}(\mathfrak{p}^-)^{-n} \otimes F(\lambda)$ and we conclude from (B.1) that X_0 acts on $\mathcal{S}(\mathfrak{p}^-)^{-n} \otimes F(\lambda)$ by $-n + \lambda(X_0)$ times the identity. Write $v = \sum_{n=0}^\infty v_0^{-n}$ according to the decomposition $N(\lambda) = \bigoplus_{n=0}^\infty \mathcal{S}(\mathfrak{p}^-)^{-n} \otimes F(\lambda)$. Since $X_0 \in i(\mathfrak{h} \cap \mathfrak{k})$, the element v_0 is annihilated by X_0 and so we must have $v_0 = v_0^{-n}$ for some $n \in \mathbb{N}_0$ with $\lambda(X_0) = n \geq 0$. But a necessary condition for $L(\lambda)$ to be unitarizable is $\lambda(X_0) < 0$ (cf. [Ne99b, Th. XI.2.37(ii)]). This gives us a contradiction and proves the lemma. ■

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